Name ______________________

Date ________________ Pd ______

## Plane Curves

Up to this point, you have been representing a graph by a single equation involving *two* variables such as $x$ and $y$. In this section, you will study situations in which it is useful to introduce a *third* variable to represent a curve in the plane.

Consider the path of an object that is propelled into the air at an angle of $45°$. If the initial velocity of the object is 48 feet per second, it can be shown that the object follows the parabolic path

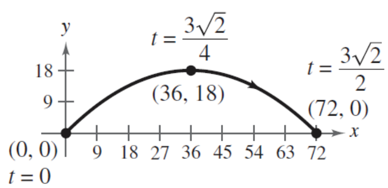
$$y = -\frac{x^2}{72} + x.$$

However, this equation does not tell the whole story. Although it does tell you *where* the object has been, it doesn't tell you *when* the object was at a given point $(x, y)$ on the path. To determine this time, you can introduce a third variable $t$, called a **parameter**. It is possible to write both $x$ and $y$ as functions of $t$ to obtain the **parametric equations** $x = 24\sqrt{2}\,t$, $y = -16t^2 + 24\sqrt{2}\,t$.

From this set of equations you can determine that at time $t = 0$, the object is at the point $(0, 0)$. Similarly, at time $t = 1$, the object is at the point $(24\sqrt{2}, 24\sqrt{2} - 16)$, and so on.

Rectangular equation:
$y = -\dfrac{x^2}{72} + x$

Parametric equations:
$x = 24\sqrt{2}t$
$y = -16t^2 + 24\sqrt{2}t$




For this particular motion problem, $x$ and $y$ are continuous functions of $t$, and the resulting path is a **plane curve**.

### Definition of a Plane Curve

If $f$ and $g$ are continuous functions of $t$ on an interval $I$, the set of ordered pairs $(f(t), g(t))$ is a **plane curve $C$**. The equations given by $x = f(t)$ and $y = g(t)$ are **parametric equations** for $C$, and $t$ is the **parameter**.

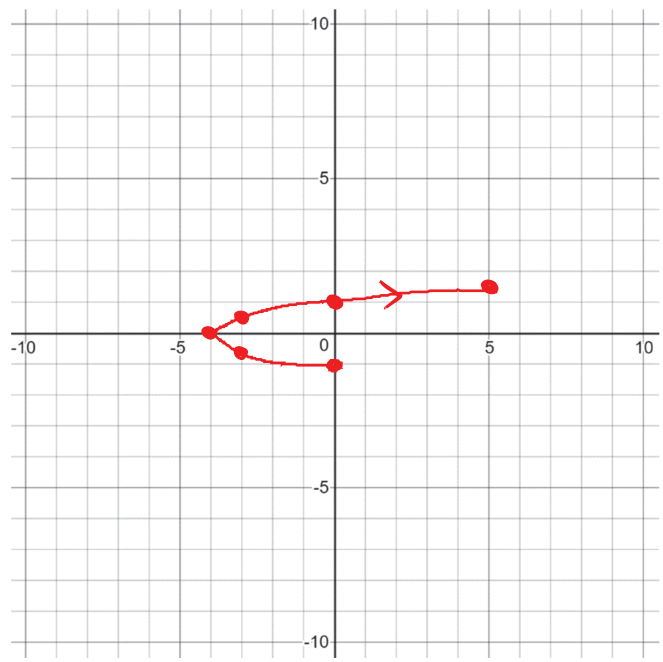
## Sketching a Plane Curve

One way to sketch a curve represented by a pair of parametric equations is to plot points in the $xy$-plane. Each set of coordinates $(x, y)$ is determined from a value chosen for the parameter $t$. By plotting the resulting points in the order of *increasing* values of $t$, you trace the curve in a specific direction. This is called the **orientation** of the curve.

## Example 1

Sketch the curve given by the parametric equations $x = t^2 - 4$, $y = t/2$, $-2 \le t \le 3$.

| $t$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|
| $x$ | $0$ | $-3$ | $-4$ | $-3$ | $0$ | $5$ |
| $y$ | $-1$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $1$ | $\frac{3}{2}$ |




Note that the graph above does not define $y$ as a function of $x$. This points out one benefit of parametric equations—they can be used to represent graphs that are more general than graphs of functions.

Two different sets of parametric equations can have the same graph. For example, the set of parametric equations $x = 4t^2 - 4$, $y = t$, $-1 \le t \le 3/2$ has the same graph as the set above. However, by comparing the values of $t$ for both sets, you can see that the second set would trace out the graph more *rapidly* (considering $t$ as time) than the first set. So, in applications, different parametric representations can be used to represent various *speeds* at which objects travel along a given path.

## Eliminating the Parameter

Many curves that are represented by sets of parametric equations have graphs that can also be represented by rectangular coordinates (in $x$ and $y$). The process of finding the rectangular equation is called **eliminating the parameter**.

| Parametric equations | Solve for $t$ in one equation. | Substitute in second equation. | Rectangular equation |
|---|---|---|---|
| $x = t^2 - 4$<br>$y = \frac{1}{2}t$ | $t = 2y$ | $x = (2y)^2 - 4$ | $x = 4y^2 - 4$ |

Now you can recognize that the equation $x = 4y^2 - 4$ represents a parabola with a horizontal axis and vertex at $(-4, 0)$.

When converting equations from parametric to rectangular form, you may need to alter the domain of the rectangular equation so that its graph matches the graph of the parametric equations.

### Example 2

Identify the curve represented by the equations $x = \dfrac{1}{\sqrt{t+1}}$ and $y = \dfrac{t}{t+1}$.

$$(t+1)\,x^2 = \frac{1}{t+1}(t+1)$$

$$(t+1)x^2 = 1$$

$$t+1 = \frac{1}{x^2}$$

$$t = \frac{1}{x^2} - 1$$

$$y = \frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} - 1 + 1}$$

$$y = \frac{\left(\frac{1}{x^2} - 1\right)x^2}{\frac{1}{x^2} \cdot x^2}$$

$$y = 1 - x^2$$

$$x > 0$$

Domain

$$t + 1 > 0$$

$$t > -1$$

### Example 3

Identify the curve represented by the equations $x = 3\cos(\theta)$ and $y = 4\sin(\theta)$, $0 \le \theta \le 2\pi$.

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

## Finding Parametric Equations for a Graph

How can you determine a set of parametric equations for a given graph or a given physical description? From the discussion following Example 1, you know that such a representation is not unique. This is further demonstrated in the following example.

### Example 4

Find a set of parametric equations to represent the graph of $y = 1 - x^2$ using the parameters (a) $t = x$ and (b) $t = 1 - x$.

$$x = t$$

$$y = 1 - t^2$$

$$x = 1 - t$$

$$y = 2t - t^2$$

$$y = 1 - (1-t)^2$$

$$= 1 - (1 - 2t + t^2)$$

$$= 1 - 1 + 2t - t^2$$