Powers of Complex Numbers

The trigonometric form of a complex number is used to raise a complex number to a power. To accomplish this, consider repeated use of the multiplication rule.

$$
\begin{aligned}
z &= r(\cos\theta + i\sin\theta) \\
z^2 &= r(\cos\theta + i\sin\theta)r(\cos\theta + i\sin\theta) = r^2(\cos 2\theta + i\sin 2\theta) \\
z^3 &= r^2(\cos 2\theta + i\sin 2\theta)r(\cos\theta + i\sin\theta) = r^3(\cos 3\theta + i\sin 3\theta) \\
z^4 &= r^4(\cos 4\theta + i\sin 4\theta) \\
z^5 &= r^5(\cos 5\theta + i\sin 5\theta) \\
&\vdots
\end{aligned}
$$

This pattern leads to **DeMoivre's Theorem**, which is named after the French mathematician Abraham DeMoivre (1667-1754).

DeMoivre's Theorem

If $z = r(\cos(\theta) + i\sin(\theta))$ is a complex number and n is a positive integer, then

$$
\begin{aligned}
z^n &= \left[r(\cos(\theta) + i\sin(\theta))\right]^n \\
&= r^n(\cos(n\theta) + i\sin(n\theta)).
\end{aligned}
$$

Example 1

Use DeMoivre's Theorem to find $(1+i)^3$. Write the answer in standard form.

Roots of Complex Numbers

Recall that a consequence of the Fundamental Theorem of Algebra is that a polynomial equation of degree n has n solutions in the complex number system. So, an equation such as $x^6 = 1$ has six solutions, and in this particular case you can find the six solutions by factoring and using the Quadratic Formula.

$$
\begin{aligned}
x^6 - 1 &= 0 \\
(x^3 - 1)(x^3 + 1) &= 0 \\
(x-1)(x^2 + x + 1)(x+1)(x^2 - x + 1) &= 0
\end{aligned}
$$

Consequently, the solutions are

$$x = \pm 1,\ \ x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i,\ \text{and}\ \ x = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

Each of these numbers is a sixth root of 1. In general, the ***n*th root of a complex number** is defined as follows.

Definition of an *n*th Root of a Complex Number

The complex number $u = a + bi$ is an ***n*th root** of the complex number z if

$$z = u^n = (a+bi)^n.$$

To find a formula for an nth root of a complex number, let u be an nth root of z, where

$$u = s(\cos(\beta) + i\sin(\beta))\ \text{and}\ z = r(\cos(\theta) + i\sin(\theta)).$$

By DeMoivre's Theorem and the fact that $u^n = z$, you have

$$s^n(\cos(n\beta) + i\sin(n\beta)) = r(\cos(\theta) + i\sin(\theta)).$$

Taking the absolute value of each side of this equation, it follows that $s^n = r$. Substituting back into the previous equation and dividing by r, you get

$$\cos(n\beta) + i\sin(n\beta) = \cos(\theta) + i\sin(\theta).$$

So, it follows that

$$\cos(n\beta) = \cos(\theta)\ \text{and}\ \sin(n\beta) = \sin(\theta).$$

Because both sine and cosine have a period of 2π, these last two equations have solutions if and only if the angles differ by a multiple of 2π. Consequently, there must exist an integer k such that

$$
\begin{aligned}
n\beta &= \theta + 2\pi k \\
\beta &= \frac{\theta + 2\pi k}{n}.
\end{aligned}
$$

By substituting this value of β into the trigonometric form of u, you get the result stated in the theorem on the following page.

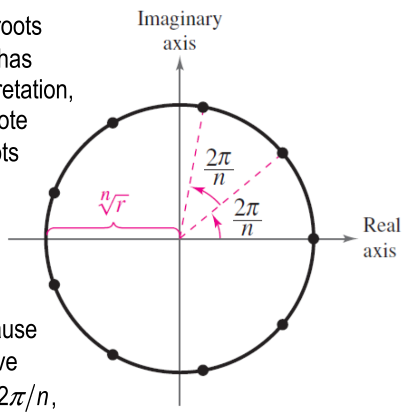
nth Roots of a Complex Number

For a positive integer n, the complex number $z = r(\cos(\theta) + i\sin(\theta))$ has exactly n distinct nth roots given by

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$$

where $k = 0, 1, 2, \ldots, n-1$.

The formula for the nth roots of a complex number z has a nice geometrical interpretation, as shown in the figure. Note that because the nth roots of z all have the same magnitude $\sqrt[n]{r}$, they all lie on a circle of radius $\sqrt[n]{r}$ with center at the origin. Furthermore, because successive nth roots have arguments that differ by $2\pi/n$, the n roots are equally spaced around the circle.



Example 2

Find the fifth roots of 1. Write your answers in standard form.

Example 3

Find the fifth roots of $128(-1+i)$. Write your answers in standard form.

Trigonometry/Precalculus

§6.5b Powers and Roots of Complex Numbers HOMEWORK

Name ______________________

Date ______________ Pd ______

In Exercises 1-4, use DeMoivre's Theorem to find the indicated power of the complex number. Write the answer in standard form.

1. $(2+2i)^6$

2. $2(\sqrt{3}+i)^5$

3. $\left[2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\right]^{12}$

4. $\left[4\left(\cos(10^\circ)+i\sin(10^\circ)\right)\right]^6$

In Exercises 5-8, find the indicated roots of the complex number. Write your answers in standard form.

5. Square roots of $5i$

6. Square roots of $1-\sqrt{3}\,i$

7. Fourth roots of $625i$

8. Cube roots of $-4\sqrt{2}(1-i)$