Name ______________________

Date ____________ Pd ______

The Dot Product of Two Vectors

So far you have studied two vector operations—vector addition and multiplication by a scalar—each of which yields another vector. In this section, you will study a third vector operation, the **dot product**. This product yields a scalar, rather than a vector.

Definition of Dot Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{0} \cdot \mathbf{v} = 0$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$

Example 1

Find the dot product of $\mathbf{u} = \langle 6, 3 \rangle$ and $\mathbf{v} = \langle 2, -4 \rangle$.

Example 2

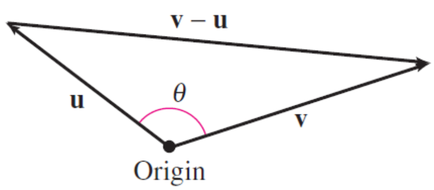
Use the vectors $\mathbf{u} = \langle 2, 2 \rangle$, $\mathbf{v} = \langle -3, 4 \rangle$, and $\mathbf{w} = \langle 1, -4 \rangle$ to find $(3\mathbf{w} \cdot \mathbf{v})\mathbf{u}$. State whether the result is a vector or a scalar.

Example 3

Use the dot product to find the magnitude of $\mathbf{u} = \langle -5, 12 \rangle$.

The Angle Between Two Vectors

The **angle between two nonzero vectors** is the angle θ, $0 \le \theta \le \pi$, between their respective standard position vectors, as shown in the figure. This angle can be found using the dot product. (Note that the angle between the zero vector and another vector is not defined.)



Angle Between Two Vectors

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}.$$

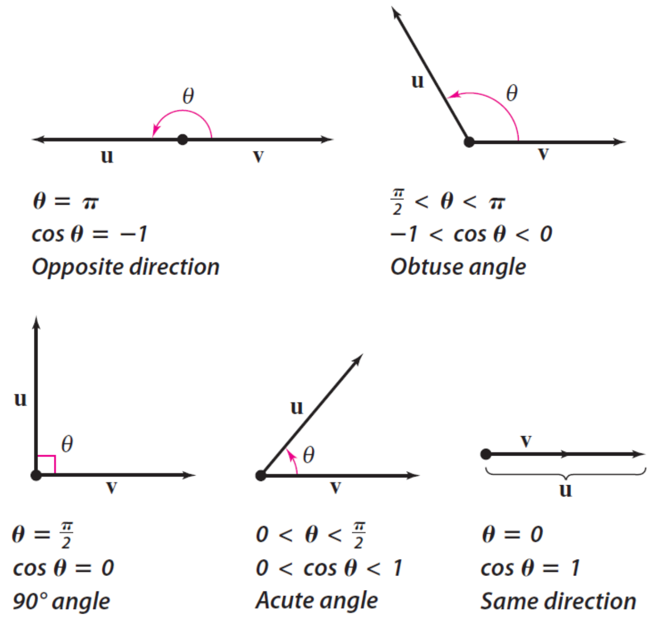
Example 4

Find the angle θ between the vectors $\mathbf{u} = \langle 4, 4 \rangle$ and $\mathbf{v} = \langle -2, 0 \rangle$.

Rewriting the expression for the angle between two vectors in the form

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta)$$

produces an alternate way to calculate the dot product. From this form, you can see that because $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are always positive, $\mathbf{u} \cdot \mathbf{v}$ and $\cos(\theta)$ will always have the same sign. The figure shows the five possible orientations of two vectors.



Definition of Orthogonal Vectors

The vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

The terms *orthogonal* and *perpendicular* mean essentially the same thing—meeting at right angles. Even though the angle between the zero vector and another vector is not defined, it is convenient to extend the definition of orthogonality to include the zero vector. In other words, the zero vector is orthogonal to every vector because $\mathbf{0} \cdot \mathbf{u} = 0$.

Example 5

Determine whether $\mathbf{u} = \frac{1}{4}(3\mathbf{i} - \mathbf{j})$ and $\mathbf{v} = 5\mathbf{i} + 6\mathbf{j}$ are orthogonal, parallel, or neither.

Name ____________________

Date ____________ Pd ______

In Exercises 1-2, find the dot product of $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u} = \langle -4,\ 1 \rangle,\ \ \mathbf{v} = \langle 2,\ -3 \rangle$

2. $\mathbf{u} = 5\mathbf{i} + \mathbf{j},\ \ \mathbf{v} = 3\mathbf{i} - \mathbf{j}$

In Exercises 3-4, use the vectors $\mathbf{u} = \langle 2,\ 2 \rangle$, $\mathbf{v} = \langle -3,\ 4 \rangle$, and $\mathbf{w} = \langle 1,\ -4 \rangle$ to find the indicated quantity. State whether the result is a vector or a scalar.

3. $4\mathbf{u} \cdot \mathbf{v}$

4. $(\mathbf{u} \cdot 2\mathbf{v})\mathbf{w}$

In Exercises 5-6, use the dot product to find the magnitude of $\mathbf{u}$.

5. $\mathbf{u} = \langle 2,\ -4 \rangle$

6. $\mathbf{u} = 6\mathbf{i} - 10\mathbf{j}$

In Exercises 7-8, find the angle θ between the vectors.

7. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$

8. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$

In Exercises 9-10, determine whether $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, parallel, or neither.

9. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = -\mathbf{i} - \mathbf{j}$

10. $\mathbf{u} = 8\mathbf{i} + 4\mathbf{j}$, $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$

In Exercises 11-12, find the value of k so that the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

11. $\mathbf{u} = 2\mathbf{i} - k\mathbf{j}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$

12. $\mathbf{u} = \mathbf{i} + 4\mathbf{j}$, $\mathbf{v} = 2k\mathbf{i} - 5\mathbf{j}$