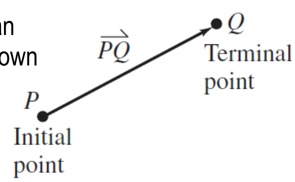
Trigonometry/Precalculus — Name ____________________

§6.3 Vectors in the Plane — Date ____________ Pd ______

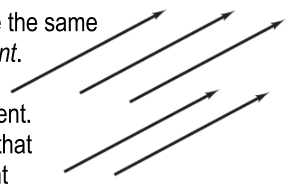
Introduction

Many quantities in geometry and physics, such as area, time, and temperature, can be represented by a single real number. Other quantities, such as force and velocity, involve both *magnitude* and *direction* and cannot be completely characterized by a single real number.

To represent such a quantity, you can use a **directed line segment**, as shown in the figure. The directed line segment $\overrightarrow{PQ}$ has **initial point** P and **terminal point** Q. Its **magnitude**, or **length**, is denoted by $\left\|\overrightarrow{PQ}\right\|$ and can be found by using the Distance Formula.



Two directed line segments that have the same magnitude and direction are *equivalent*. For example, the directed line segments in the figure are all equivalent. The set of all directed line segments that are equivalent to a given line segment $\overrightarrow{PQ}$ is a **vector v in the plane**, written $\mathbf{v} = \overrightarrow{PQ}$. Vectors are denoted by lowercase, boldcase letters such as **u**, **v**, and **w**.



Component Form of a Vector

The directed line segment whose initial point is the origin is often the most convenient representative of a set of equivalent directed line segments. This representative of the vector **v** is in **standard position**.

A vector whose initial point is at the origin $(0, 0)$ can be uniquely represented by the coordinates of its terminal point (v_1, v_2). This is the **component form of a vector v**, written as

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

The coordinates v_1 and v_2 are the *components* of **v**. If both the initial point and the terminal point lie at the origin, **v** is the **zero vector** and is denoted by $\mathbf{0} = \langle 0, 0 \rangle$.

Component Form of a Vector

The component form of a vector with initial point $P = (p_1, p_2)$ and terminal point $Q = (q_1, q_2)$ is given by

$$\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = \mathbf{v}.$$

The magnitude (or length) of **v** is given by

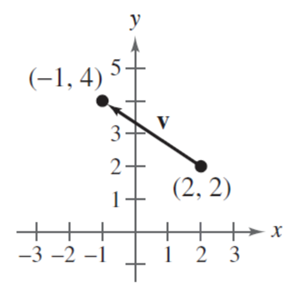
$$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}.$$

If $\|\mathbf{v}\| = 1$, **v** is a **unit vector**. Moreover, $\|\mathbf{v}\| = 0$ if and only if **v** is the zero vector **0**.

Two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are *equal* if and only if $u_1 = v_1$ and $u_2 = v_2$.

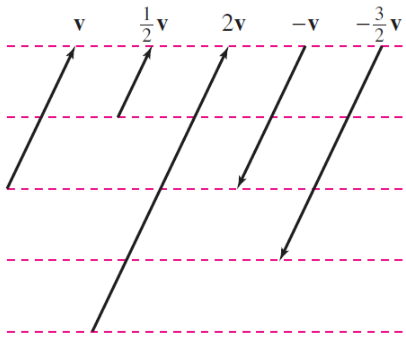
Example 1

Find the component form and the magnitude of the vector **v**.

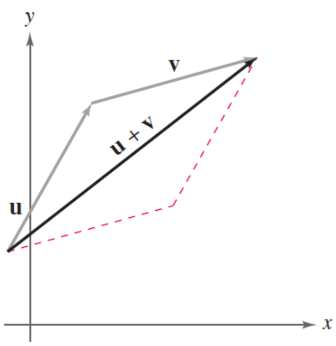


Vector Operations

The two basic vector operations are **scalar multiplication** and **vector addition**. Geometrically, the product of a vector $\mathbf{v}$ and a scalar k is the vector that is $|k|$ times as long as $\mathbf{v}$. If $k > 0$, $k\mathbf{v}$ has the same direction as $\mathbf{v}$; if $k < 0$, $k\mathbf{v}$ has the opposite direction of $\mathbf{v}$, as shown in the figure.



To add two vectors $\mathbf{u}$ and $\mathbf{v}$ geometrically, first position them (without changing their lengths or directions) so that the initial point of the second vector $\mathbf{v}$ coincides with the terminal point of the first vector $\mathbf{u}$. The sum $\mathbf{u}+\mathbf{v}$ is the vector formed by joining the initial point of the first vector $\mathbf{u}$ with the terminal point of the second vector $\mathbf{v}$, as shown in the figure.



Definition of Vector Addition and Scalar Multiplication

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let k be a scalar (a real number). Then the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u}+\mathbf{v} = \langle u_1 + v_1,\ u_2 + v_2 \rangle$$

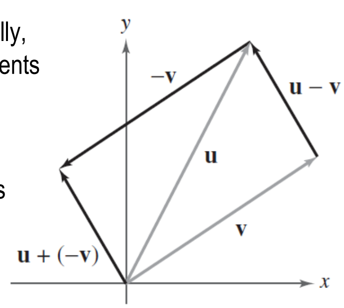
and the **scalar multiple** of k times $\mathbf{u}$ is the vector

$$k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle.$$

The **negative** of $\mathbf{v} = \langle v_1, v_2 \rangle$ is $-\mathbf{v} = (-1)\mathbf{v} = \langle -v_1, -v_2 \rangle$ and the **difference** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{aligned} \mathbf{u}-\mathbf{v} &= \mathbf{u}+(-\mathbf{v}) \\ &= \langle u_1 - v_1,\ u_2 - v_2 \rangle. \end{aligned}$$

To represent $\mathbf{u}-\mathbf{v}$ geometrically, you can use directed line segments with the *same* initial point. The difference $\mathbf{u}-\mathbf{v}$ is the vector from the terminal point of $\mathbf{v}$ to the terminal point of $\mathbf{u}$, which is equal to $\mathbf{u}+(-\mathbf{v})$, as shown in the figure.



Example 2

Given $\mathbf{u} = \langle 4, 2 \rangle$ and $\mathbf{v} = \langle 7, 1 \rangle$, find $2\mathbf{u} - 3\mathbf{v}$.

Properties of Vector Addition and Scalar Multiplication

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors and let c and d be scalars. Then the following properties are true.

1. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$
3. $\mathbf{u}+\mathbf{0} = \mathbf{u}$
4. $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$
5. $c(d\mathbf{u}) = (cd)\mathbf{u}$
6. $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$
7. $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$
8. $1(\mathbf{u}) = \mathbf{u}$, $0(\mathbf{u}) = \mathbf{0}$
9. $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$

Unit Vectors

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given nonzero vector $\mathbf{v}$. To do this, you can divide $\mathbf{v}$ by its length to obtain

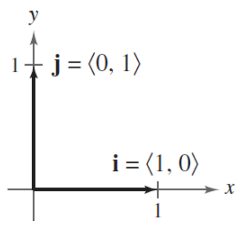
$$\mathbf{u} = \text{unit vector} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

Note that $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$. The vector $\mathbf{u}$ has a magnitude of 1 and the same direction as $\mathbf{v}$. The vector $\mathbf{u}$ is called a **unit vector in the direction of $\mathbf{v}$**.

Example 3

Find a unit vector in the direction of $\mathbf{v} = \langle -1, 1 \rangle$.

The unit vectors $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are called the **standard unit vectors** and are denoted by $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ as shown in the figure. These vectors can be used to represent any vector $\mathbf{v} = \langle v_1, v_2 \rangle$ as follows.



$$\begin{aligned}\mathbf{v} &= \langle v_1, v_2 \rangle \\ &= v_1\langle 1, 0 \rangle + v_2\langle 0, 1 \rangle \\ &= v_1\mathbf{i} + v_2\mathbf{j}\end{aligned}$$

The scalars v_1 and v_2 are called the **horizontal and vertical components of v**, respectively. The vector sum $v_1\mathbf{i} + v_2\mathbf{j}$ is called a **linear combination** of the vectors $\mathbf{i}$ and $\mathbf{j}$. Any vector in the plane can be written as a linear combination of the standard unit vectors $\mathbf{i}$ and $\mathbf{j}$.

Example 4

Given: The initial point $(-3, 1)$ and terminal point $(4, 5)$ of a vector. Write the vector as a linear combination of the standard unit vectors $\mathbf{i}$ and $\mathbf{j}$.

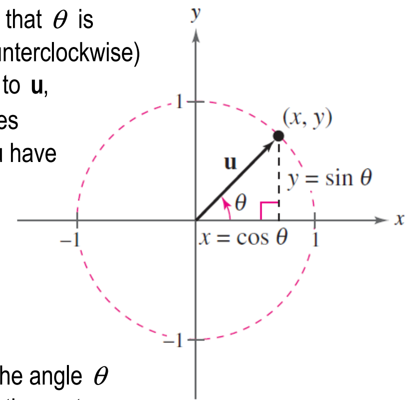
Example 5

Given $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j}$, find the component form of $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$.

Direction Angles

If $\mathbf{u}$ is a *unit vector* such that θ is the angle (measured counterclockwise) from the positive x-axis to $\mathbf{u}$, the terminal point of $\mathbf{u}$ lies on the unit circle and you have

$$\begin{aligned}\mathbf{u} &= \langle x, y \rangle \\ &= \langle \cos(\theta), \sin(\theta) \rangle \\ &= \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}\end{aligned}$$



as shown in the figure. The angle θ is the **direction angle** of the vector $\mathbf{u}$.

Suppose that $\mathbf{u}$ is a unit vector with direction angle θ. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is any vector that makes an angle θ with the positive x-axis, then it has the same direction as $\mathbf{u}$ and you can write

$$\begin{aligned}\mathbf{v} &= \|\mathbf{v}\|\langle \cos(\theta), \sin(\theta) \rangle \\ &= \|\mathbf{v}\|\cos(\theta)\mathbf{i} + \|\mathbf{v}\|\sin(\theta)\mathbf{j}.\end{aligned}$$

Because $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \|\mathbf{v}\|\cos(\theta)\mathbf{i} + \|\mathbf{v}\|\sin(\theta)\mathbf{j}$, it follows that the direction angle θ for $\mathbf{v}$ is determined from

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\|\mathbf{v}\|\sin(\theta)}{\|\mathbf{v}\|\cos(\theta)} = \frac{b}{a}.$$

Example 6

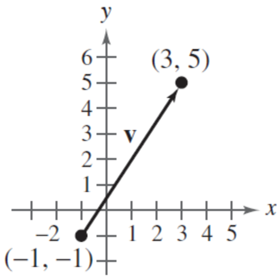
Find the magnitude and direction angle of the vector $\mathbf{v} = -2\mathbf{i} + 5\mathbf{j}$.

Trigonometry/Precalculus Name ____________________

§6.3 Vectors in the Plane HOMEWORK Date ______________ Pd ______

In Exercises 1-2, find the component form and the magnitude of the vector $\mathbf{v}$.

1.



2. Initial point $(5/2,\ -2)$ and terminal point $(1,\ 2/5)$

In Exercises 3-4, find (a) $\mathbf{u}+\mathbf{v}$ and (b) $2\mathbf{u}-3\mathbf{v}$.

3. $\mathbf{u}=\langle -6,\ -8\rangle$ and $\mathbf{v}=\langle 2,\ 4\rangle$

4. $\mathbf{u}=2\mathbf{i}-\mathbf{j}$ and $\mathbf{v}=-\mathbf{i}+\mathbf{j}$

In Exercises 5-6, find a unit vector in the direction of the given vector.

5. $\mathbf{v}=\langle -24,\ -7\rangle$

6. $\mathbf{v}=4\mathbf{i}-3\mathbf{j}$

Trigonometry/Precalculus

§6.3 Vectors in the Plane HOMEWORK

Name ______________________

Date ______________ Pd ______

In Exercises 7-8, the initial and terminal points of a vector are given. Write the vector as a linear combination of the standard unit vectors $\mathbf{i}$ and $\mathbf{j}$.

7. Initial point $(-1,\ -5)$ and terminal point $(2,\ 3)$

8. Initial point $(-6,\ 4)$ and terminal point $(0,\ 1)$

In Exercises 9-10, find the component form of $\mathbf{v}$, where $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j}$.

9. $\mathbf{v} = -\mathbf{u} + \mathbf{w}$

10. $\mathbf{v} = \frac{1}{2}(3\mathbf{u} + \mathbf{w})$

In Exercises 11-12, find the magnitude and direction angle of the vector $\mathbf{v}$.

11. $\mathbf{v} = -4\mathbf{i} - 7\mathbf{j}$

12. $\mathbf{v} = 12\mathbf{i} + 15\mathbf{j}$