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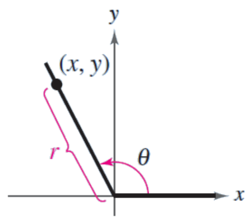
§4.4 Trigonometric Functions of Any Angle — Date ______________ Pd ______

In §4.3, the definitions of trigonometric functions were restricted to acute angles. In this section, the definitions are extended to cover *any* angle. If θ is an *acute* angle, the definitions here coincide with those given in the preceding section.

Definitions of Trigonometric Functions of Any Angle

Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.



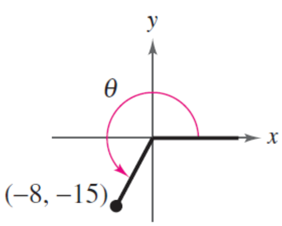
$\sin(\theta) = \dfrac{y}{r}$ $\qquad$ $\cos(\theta) = \dfrac{x}{r}$

$\tan(\theta) = \dfrac{y}{x},\ x \neq 0$ $\qquad$ $\cot(\theta) = \dfrac{x}{y},\ y \neq 0$

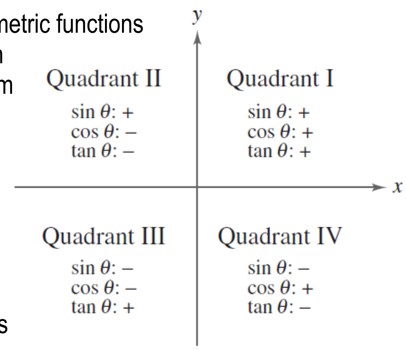
$\sec(\theta) = \dfrac{r}{x},\ x \neq 0$ $\qquad$ $\csc(\theta) = \dfrac{r}{y},\ y \neq 0$

Example 1

Determine the exact values of the six trigonometric functions of the angle θ.



The *signs* of the trigonometric functions in the four quadrants can be determined easily from the definitions of the functions. For instance, because $\cos(\theta) = x/r$, it follows that $\cos(\theta)$ is positive wherever $x > 0$, which is in Quadrants I and IV. (Remember, r is always positive.)



Example 2

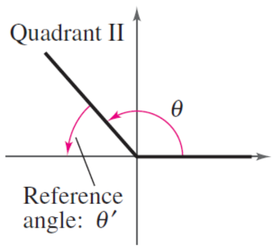
Given $\tan(\theta) = -15/8$ and $\sin(\theta) < 0$, find the values of the six trigonometric functions of θ.

Reference Angles

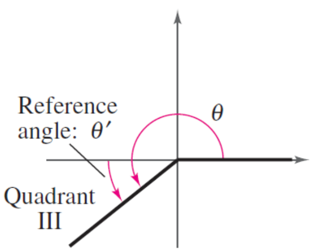
The values of the trigonometric functions of angles greater than 90° (or less than 0°) can be determined from their values at corresponding acute angles called **reference angles**.

Definition of Reference Angle

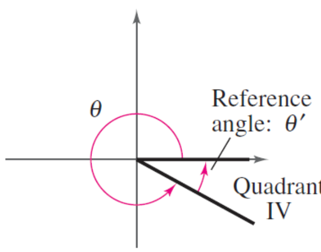
Let θ be an angle in standard position. Its **reference angle** is the acute angle θ' formed by the terminal side of θ and the horizontal axis.



$\theta' = \pi - \theta$ (radians)
$\theta' = 180^\circ - \theta$ (degrees)



$\theta' = \theta - \pi$ (radians)
$\theta' = \theta - 180^\circ$ (degrees)



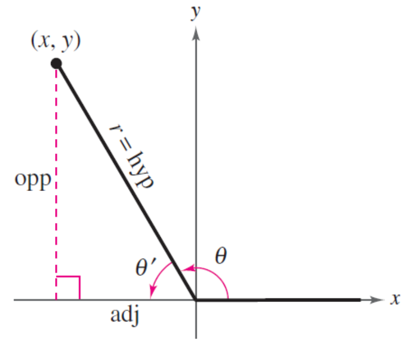
$\theta' = 2\pi - \theta$ (radians)
$\theta' = 360^\circ - \theta$ (degrees)

Example 3

Given $\theta = \dfrac{3\pi}{4}$, find the reference angle θ' and sketch θ and θ' in standard position.

Trigonometric Functions of Real Numbers

To see how a reference angle is used to evaluate a trigonometric function, consider the point (x, y) on the terminal side of θ, as shown in the figure.



By definition, you know that $\sin(\theta) = \dfrac{y}{r}$ and $\tan(\theta) = \dfrac{y}{x}$.

For the right triangle with acute angle θ' and sides of length $|x|$ and $|y|$, you have

$$\sin(\theta') = \frac{\text{opp}}{\text{hyp}} = \frac{|y|}{r} \text{ and } \tan(\theta') = \frac{\text{opp}}{\text{adj}} = \frac{|y|}{|x|}.$$

So, it follows that $\sin(\theta)$ and $\sin(\theta')$ are equal, *except possibly in sign*. The same is true for $\tan(\theta)$ and $\tan(\theta')$ and for the other four trigonometric functions. In all cases, the sign of the function value can be determined by the quadrant in which θ lies.

Evaluating Trigonometric Functions of Any Angle

To find the value of a trigonometric function of any angle θ:

1. Determine the function value for the associated reference angle θ'.
2. Depending on the quadrant in which θ lies, affix the appropriate sign to the function value.

By using reference angles and the special angles discussed in the previous section, you can greatly extend the scope of *exact* trigonometric values. For instance, knowing the function values of 30° means that you know the function values of all angles for which 30° is a reference angle.

θ (deg)	0°	30°	45°	60°	90°
θ (rad)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef

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Example 4

Evaluate the sine, cosine, and tangent of the angle $\theta = -\frac{17\pi}{6}$ without using a calculator.

Example 5

Given $\sin(\theta) = -\frac{3}{5}$ and θ is in Quadrant IV, find $\cos(\theta)$.

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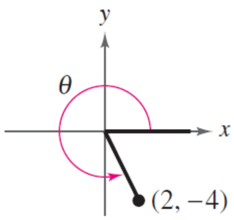
§4.4 Trigonometric Functions of Any Angle HOMEWORK | Date ______________ Pd ______

1. Determine the exact values of the six trigonometric functions of the angle θ.



In Exercises 2-3, find the values of the six trigonometric functions of θ.

2. $\cos(\theta) = -4/5$, θ lies in Quadrant III

3. $\sec(\theta) = -2$, $0 \le \theta \le \pi$

In Exercises 4-5, find the reference angle θ' and sketch θ and θ' in standard position.

4. $\theta = \dfrac{5\pi}{3}$

5. $\theta = -\dfrac{5\pi}{6}$

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In Exercises 6-8, evaluate the sine, cosine, and tangent of the angle without using a calculator.

6. $\frac{5\pi}{3}$

7. $\frac{3\pi}{4}$

8. $-\frac{\pi}{6}$

In Exercises 9-11, find the indicated trigonometric value in the specified quadrant.

9. Find $\sin(\theta)$, given $\cot(\theta) = -3$ and θ is in Quadrant II

10. Find $\tan(\theta)$, given $\sec(\theta) = -9/4$ and θ is in Quadrant III

11. Find $\csc(\theta)$, given $\tan(\theta) = -5/4$ and θ is in Quadrant IV