Name ______________________

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Introduction to Rational Functions

A **rational function** can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

In general, the *domain* of a rational function of x includes all real numbers except x-values that make the denominator zero. Much of the discussion of rational functions will focus on their graphical behavior near these x-values.

Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$, either from the right or from the left.
2. The line $y = b$ is a **horizontal asymptote** of the graph of f if $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$.

Vertical and Horizontal Asymptotes of a Rational Function

Let f be the rational function

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of $D(x)$.
2. The graph of f has at most one *horizontal* asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.

- If $\boxed{n < m}$, the graph of f has the line $\boxed{y = 0}$ (the x-axis) as a horizontal asymptote.
- If $\boxed{n = m}$, the graph of f has the line $\boxed{y = a_n / b_m}$ as a horizontal asymptote, where a_n is the leading coefficient of the numerator and b_m is the leading coefficient of the denominator.
- If $\boxed{n > m}$, the graph of f has $\boxed{\text{no horizontal asymptote.}}$

Example 1

Find all horizontal and vertical asymptotes of the graph of the rational function.

(a) $f(x) = \dfrac{2x^1}{3x^2 + 1}$

$n = 1$

$m = 2$

$n < m$

HA: $y = 0$

VA: none

$3x^2 + 1 = 0$

$-1 \quad -1$

$\dfrac{3x^2}{3} = \dfrac{-1}{3}$

$x^2 = -\dfrac{1}{3}$

no real solution

(b) $f(x) = \dfrac{2x^2}{1x^2 - 1}$

$n = 2$

$m = 2$

$n = m$

$y = \dfrac{2}{1}$

HA: $y = 2$

VA: $x = 1$, $x = -1$

$x^2 - 1 = 0$

$+1 \quad +1$

$x^2 = 1$

"missing step" $|x| = 1$

$x = \pm 1$

Values for which a rational function is undefined (the denominator is zero) results in a vertical asymptote or a **hole** in the graph.

Example 2

Find all horizontal and vertical asymptotes and holes in the graph of $f(x)=\frac{x^2+x-2}{x^2-x-6}$. $=\frac{\cancel{(x+2)}(x-1)}{\cancel{(x+2)}(x-3)}=\frac{x-1}{x-3},\ x\neq -2$

$\frac{-2-1}{-2-3}=\frac{-3}{-5}=\frac{3}{5}$

hole: $(-2, \frac{3}{5})$

$n=2,\ m=2,\ n=m$

HA: $y=1$

$y=\frac{1}{1}=1$

VA: $x=3$

$x-3=0$
$+3\ \ +3$
$x=3$

A function that is not rational can have two horizontal asymptotes—one to the left and one to the right.

$|a|=\begin{cases} a & \text{if } a\geq 0 \\ -a & \text{if } a<0 \end{cases}$

Example 3

Find all horizontal and vertical asymptotes and holes in the graph of $f(x)=\frac{x+10}{|x|+2}$.

$=\begin{cases} \frac{x+10}{x+2} & \text{if } x\geq 0 \\ \frac{x+10}{-x+2} & \text{if } x<0 \end{cases}$

hole: none

HA: $y=1$, $y=-1$

VA: none

for $x\geq 0$

$y=\frac{1}{1}=1$

$x+2=0$
$-2\ -2$
$x=-2$

for $x<0$

$y=\frac{1}{-1}=-1$

$-x+2=0$
$+x\ \ \ \ +x$
$2=x$

5. $f(x)=\dfrac{x^2-25}{x^2+5x} = \dfrac{(x+5)(x-5)}{x(x+5)} = \dfrac{x-5}{x},\ x \neq -5$

$\dfrac{-5-5}{-5} = \dfrac{-10}{-5} = 2$

hole: $(-5, 2)$

HA: $y=1$

VA: $x=0$

$n=2,\ m=2$

$n=m$

$y=\frac{1}{1}=1$

$x=0$

7. $f(x)=\dfrac{x-3}{|x|} = \begin{cases} \dfrac{x-3}{x}, & x \geq 0 \\ \dfrac{x-3}{-x}, & x<0 \end{cases}$

for $x \geq 0$	for $x<0$
$n=1,\ m=1$	$n=1,\ m=1$
$n=m$	$n=m$
$y=\frac{1}{1}=1$	$y=\frac{1}{-1}=-1$
$x=0$	$-x=0$
	$x=0$

hole: none

HA: $y=1,\ y=-1$

VA: $x=0$

6. $f(x)=\dfrac{2x+3}{|x|-2} = \begin{cases} \dfrac{2x+3}{x-2}, & x \geq 0 \\ \dfrac{2x+3}{-x-2}, & x<0 \end{cases}$

for $x \geq 0$	for $x<0$
$n=1,\ m=1$	$n=1,\ m=1$
$n=m$	$n=m$
$y=\frac{2}{1}=2$	$y=\frac{2}{-1}=-2$
$x-2=0$	$-x-2=0$
$x=2$	$-2=x$

hole: none

HA: $y=2,\ y=-2$

VA: $x=2,\ x=-2$

8. $f(x)=\dfrac{1}{|x|+1} = \begin{cases} \dfrac{1}{x+1}, & x \geq 0 \\ \dfrac{1}{-x+1}, & x<0 \end{cases}$

for $x \geq 0$	for $x<0$
$n=0,\ m=1$	$n=0,\ m=1$
$n<m$	$n<m$
$x+1=0$	$-x+1=0$
$x=-1$	$1=x$
(not in $x \geq 0$)	(not in $x<0$)

hole: none

HA: $y=0$

VA: none