

Introduction to Rational Functions

A **rational function** can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

In general, the *domain* of a rational function of x includes all real numbers except x -values that make the denominator zero. Much of the discussion of rational functions will focus on their graphical behavior near these x -values.

Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f if $f(x) \rightarrow \infty$ or $f(x) \rightarrow -\infty$ as $x \rightarrow a$, either from the right or from the left.
2. The line $y = b$ is a **horizontal asymptote** of the graph of f if $f(x) \rightarrow b$ as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

Vertical and Horizontal Asymptotes of a Rational Function

Let f be the rational function

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has *vertical asymptotes* at the zeros of $D(x)$.
2. The graph of f has at most one *horizontal asymptote* determined by comparing the degrees of $N(x)$ and $D(x)$.
 - If $n < m$, the graph of f has the line $y = 0$ (the x -axis) as a horizontal asymptote.
 - If $n = m$, the graph of f has the line $y = a_n/b_m$ as a horizontal asymptote, where a_n is the leading coefficient of the numerator and b_m is the leading coefficient of the denominator.
 - If $n > m$, the graph of f has no horizontal asymptote.

Example 1

Find all horizontal and vertical asymptotes of the graph of the rational function.

(a) $f(x) = \frac{2x}{3x^2 + 1}$

(b) $f(x) = \frac{2x^2}{x^2 - 1}$

Values for which a rational function is undefined (the denominator is zero) results in a vertical asymptote or a **hole** in the graph.

Example 2

Find all horizontal and vertical asymptotes and holes in the graph

of $f(x) = \frac{x^2 + x - 2}{x^2 - x - 6}$.

A function that is not rational can have two horizontal asymptotes—one to the left and one to the right.

Example 3

Find all horizontal and vertical asymptotes and holes in the graph

of $f(x) = \frac{x + 10}{|x| + 2}$.

In Exercises 1-8, find all horizontal and vertical asymptotes and holes in the graph of $f(x)$.

1. $f(x) = \frac{2}{x+2}$

2. $f(x) = \frac{1-x}{x}$

3. $f(x) = \frac{2x+x^2}{2x-x^2}$

4. $f(x) = \frac{x^2+2x+1}{2x^2-x-3}$

5. $f(x) = \frac{x^2 - 25}{x^2 + 5x}$

7. $f(x) = \frac{x-3}{|x|}$

6. $f(x) = \frac{2x+3}{|x|-2}$

8. $f(x) = \frac{1}{|x|+1}$