Advanced Placement Calculus: Differences within University Instruction

Jason Nagaoka

Hawai‘i Pacific University

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Student Name: Jason Nagaoka

Title of Professional Paper: Advanced Placement Calculus: Differences within University Instruction

E.S. Van Gorder, Ph.D.
Associate Professor of Education

Date

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Abstract

For more than a half-century, students across the world have had the opportunity to participate in a program that claims to promote college-level academic rigor in the secondary school classroom. This program, offered by the College Board, is the Advanced Placement (AP) program, and what started with meager beginnings has now grown into an offering of more than 30 courses and examinations in over 17,000 schools worldwide. One of the subjects most often studied under the AP banner is calculus, which is offered in one of two varieties—AP Calculus AB and AP Calculus BC (the latter of which is an extension of the former). Throughout this study, "AP Calculus" will refer to both courses together, as they share many commonalities; where the two courses require distinction, the designations "Calculus AB" and "Calculus BC" will be used.

The focus of this thesis was to examine the AP Calculus program as it is currently established, and to analyze where there may be potential shortcomings. This was done through independent research into 50 different institutions of varying prestige, examining the curricula and philosophies of these institutions' Calculus I courses for comparison with the Calculus AB course. Through this research, it was desired that improved understanding of the similarities and differences between AP Calculus and college calculus would lead to teachers being better prepared to bridge that divide for their students.

Importance of the Problem

The Problem

In its latest annual report, the College Board (2012c) indicated that AP Calculus is as popular as ever, with 267,722 high school graduates from the class of 2011 taking either of the two AP Calculus examinations at some point in their high school career; this total was eclipsed only by the number of graduates taking AP US History, AP English Language and Composition, and AP English Literature and Composition. In addition, at many—if not most—universities, students who perform well on their AP Calculus examination have the option of receiving college credit and/or advanced placement in the college calculus sequence. For those taking the Calculus AB examination, this typically means earning credit for or a waiver from the Calculus I course, allowing the student to enter Calculus II as soon as their first term.

With the growing popularity of AP Calculus, however, come the growing concerns over whether these students are receiving an educational experience equivalent to their counterparts in college who did not take AP Calculus. It should be noted that some students enter AP Calculus without the prerequisite knowledge and skills needed in order to succeed, due to a desire to have the “AP Calculus” label on high school transcripts when applying for college admissions. It should also be noted that while the AP Calculus curriculum is relatively common across all high schools and is aligned to most college curricula, there might be some differences between college institutions; therefore, students at some universities may be learning more or less than those who took the AP Calculus course. Finally, for those that opt to receive advanced placement into subsequent college courses, there are concerns that these students may have difficulty transitioning from the expectations of the AP Calculus program to those of their collegiate program, due to differences in philosophy.

Purpose of the Research

The purpose of this study is to identify how the practices and philosophy of the AP Calculus program are different from calculus programs offered at the college and university level. With more and more high school students taking AP Calculus and jumping straight into college calculus courses, it is imperative that AP Calculus instructors understand what needs to be done to better prepare their students not only for the culminating AP Calculus examination, but also for future experiences at the undergraduate level.

Where differences may abound, the research is intended to educate, inform, and advise, giving AP Calculus instructors an additional resource to consult with when faced with decisions about adjustments to curriculum. While this research is not intended to be exhaustive, it is desired that it will still be a valuable resource for the community of AP Calculus teachers, many of whom share their own expertise in workshops and electronic discussion groups. Whereas many of those discussions are focused solely on AP Calculus curriculum, however, this research is intended to broaden the scope of understanding toward the bigger picture of student education after AP Calculus and/or Calculus I.

Review of Relevant Literature

About AP Calculus: Curriculum

According to the College Board (2012b), Calculus AB is a yearlong course in differential and integral calculus; while each individual college or university creates its own policies regarding placement and credit for performance on the Calculus AB examination, the course is typically equivalent to the first semester of a college two-semester single-variable calculus sequence. Calculus BC is also a yearlong course, which includes all of the content from Calculus AB as well as other subjects from single-variable calculus; this course is typically equivalent to the full college two-semester single-variable calculus sequence.

Prior to taking AP Calculus, students are expected to have completed what the College Board (2012b) called “the equivalent of four full years of high school mathematics.” As differentiated by the Hawai‘i public high school curriculum, this matches with one year of Algebra I, one year of Geometry, one year of Algebra II, one semester of Trigonometry, and one semester of either Analytic Geometry or Precalculus (depending on the high school). This may mean taking Algebra I in the 8th grade and Geometry in the 9th grade, although it is sometimes possible to take Algebra I in the 9th grade and Geometry during summer school, prior to the start of the 10th grade year.

The content of the Calculus AB curriculum is centered on three topics: (1) functions, graphs, and limits; (2) derivatives; and (3) integrals. The Calculus BC curriculum encompasses and expands upon the entire Calculus AB curriculum, plus the key addition of a fourth topic: (4) polynomial approximations and series. A topic outline of the Calculus BC course (which also encompasses the AB outline) is included at the end of this report in Appendix A (College Board, 2012b).

In a short historical piece on AP Calculus, Bressoud (2010a) noted that “most” colleges’ first semester calculus course cover differential calculus, with a brief introduction to integration; meanwhile, the Calculus AB curriculum covers additional topics of integration, including various types of applications. His claim was that at these institutions, the assumption is students will continue and take the second semester of calculus; meanwhile, it is not assumed that students who take AP Calculus will continue with further calculus studies at the university level, so the AP program offers a broader curriculum than their collegiate counterparts do. In a bit of a contradiction, however, Bressoud (2010b) later noted that any content that is not typically covered or required for a first semester calculus course should not be included in the Calculus AB curriculum, which relegated some topics such as l’Hôpital’s Rule and antidifferentiation by parts solely to the Calculus BC curriculum.

About AP Calculus: Examination

Students enrolled in AP Calculus courses typically take one of the two AP Calculus examinations offered annually in May. One examination covers Calculus AB topics, while the other covers Calculus BC topics. Since both examinations are offered concurrently, students must choose which of the tests is most appropriate for them. It is not required for students enrolled in an AP Calculus course to take the AP Calculus examination; also, current enrollment in an AP Calculus course is not a requirement for taking an AP Calculus examination. The examination itself is written each year by a committee of college instructors and expert high school AP Calculus teachers (College Board, 2012b).

The AP Calculus examination is three hours, 15 minutes long. There are 45 multiple-choice questions, where answers are marked either as correct (one point) or incorrect (zero points), with no penalty for guessing. There are also six free-response questions, worth nine points each, with partial credit given according to a grading rubric specified for each problem. Both the multiple-choice and free-response sections of the AP Calculus examination are given equal weight in the determination of the final score; thus, the multiple-choice score is multiplied by 1.2, so that the maximum possible points matches that from the free-response section. The two scores are then totaled to determine a student's composite score (College Board, 2012b).

Each composite score is then converted to an AP examination score on the 1-5 range, where 5 means "extremely well qualified," 4 means "well qualified," 3 means "qualified," 2 means "possibly qualified," and 1 means "no recommendation" (College Board, 2012b). An AP examination score of 5 is supposedly equivalent to an A in a similar college course, an AP examination score of 4 is supposedly equivalent to an A–, B+, or B, and an AP examination score of 3 is supposedly equivalent to a B–, C+, or C. In order to ensure that examination scores are accurate representations of student understanding from year to year, the examination developers add some multiple-choice questions that were given in previous examinations, in order to properly norm-reference (Gollub, Bertenthal, Labov, & Curtis, 2002).

About AP Calculus: Philosophy

The 1990s brought about shifts in philosophy that led to changes in the AP Calculus curriculum. According to Bressoud (2010b), the first key factor was a re-emphasis on conceptual understanding of calculus topics, rather than a focus on procedures and manipulative skills. Another key factor that led to philosophical change was the proliferation of graphing calculators in the schools.

As the College Board was interested in truly "developing the students' understanding of the concepts of calculus" (2012b), it utilized the so-called "Rule of Four" to emphasize the roles of graphical, numerical, analytical, and verbal analysis and representations. Rather than only focusing on "manipulation (and) memorization of functions, curves, theorems, (and) problem types," students are now required to understand calculus concepts when displayed in various ways. For example, students might have to use the graph of $f'(x)$ in order to describe the behavior of $f(x)$, without being given an expression for either function. Similarly, students might have to utilize tabular data to show their understanding of concepts such as the Intermediate Value Theorem or Mean Value Theorem, rather than spitting out a memorized "proof" based on a given function.

Because the AP Calculus curriculum is focused on teaching understanding in a way that might be different from college curricula, publishers began to gear specific textbooks toward use in AP Calculus classes. Some, such as *Calculus: Graphical, Numerical, Algebraic* (4th ed.) by Finney, Demana, Waits, & Kennedy (2012), even went so far as to mention the points of emphasis from the Rule of Four in the title of the book. Some textbooks widely used by colleges also have special versions geared specifically for the AP Calculus classroom, such as *Rogawski's Calculus for AP* (2nd ed.) by Rogawski & Cannon (2012).

In order to support efforts to promote learning through graphical and numerical methods, the College Board mandated the use of graphing calculators in the AP Calculus curriculum. While such technology is sometimes used just to "confirm written work" (i.e., to check calculations), the AP Calculus curriculum instead emphasized that graphing calculators should be used in order to "reinforce the relationships among the multiple representations of functions … to implement experimentation, and to assist in interpreting results" (College Board, 2012b). In other words, graphing calculator technology now was meant to be used not only to guide students in their conceptual understanding of calculus topics, but also to help in interpreting results that came about.

In order to support efforts to promote learning through verbal methods, the AP Calculus curriculum was developed with a stated goal that "students should be able to communicate mathematics and explain solutions to problems both verbally and in written sentences" (College Board, 2012b). While the AP Calculus examination does not include a section on verbal explanations, the written free-response section requires students to be able to explain their understanding. While they may be able to perform the calculations with ease (even on a graphing calculator), they must also be able to interpret these results through written words, so that someone else who reads their work can clearly see that they understand the concepts being taught.

About AP Calculus: Critiques and Criticisms

A natural concern of the growth of AP Calculus is whether or not results from the AP Calculus examinations are aligned with results from college Calculus I courses. Bressoud (2010c) cited recent studies at the University of California, San Diego and the University of Illinois at Urbana-Champaign that indicated "students with a 4 or higher on the AB examination … are at least as well prepared for mainstream Calculus II as the students who have taken Calculus I at that university." Considering a score of 4+ on the Calculus AB examination equates roughly to a college grade of B or higher (College Board, 2012b), this indicates that roughly 35.6 percent of students who have taken the Calculus AB examination are as well-prepared as their collegiate counterparts (College Board, 2012c). In contrast, the preliminary results cited by Bressoud (2011a) indicated that 50 percent of students in Calculus I earned a B or higher in that course when taken at the post-secondary level. It is possible this indicates that the Calculus AB program does not prepare its students as well as a collegiate program should, although this hypothesis should be balanced by the realization that the grades of the 14,000+ students in the study cited by Bressoud were not necessarily determined with a standard rubric or scale, and standards may differ greatly between institutions.

Another study by Keng & Dodd (2008) looked at University of Texas at Austin students from 1998 to 2001, dividing them into groups and looking for significant differences in performance at the $\alpha = 0.05$ level. In examining student performance in Calculus II, the researchers noted that in two of the four years, students who received advanced placement from their Calculus AB examination results (allowing them to skip Calculus I) did better in Calculus II than those who did not take AP Calculus at all. This indicated some evidence that the Calculus AB examination successfully identified the students that were well prepared enough to advance into the next course. It should also be noted that in all four years, those who took the Calculus AB examination but did not perform well enough to earn credit did worse in Calculus II than their counterparts who did not take AP Calculus at all in high school, despite the fact that both groups still took Calculus I in college. This could provide some evidence to back the claim that students should not be enrolled in AP Calculus courses if they are not adequately prepared beforehand to succeed to the point of earning college credit.

Bressoud (2010c) noted that in his conversations with high school teachers, many had expressed concerns that students “do not have the time to develop understanding” because so much time is used on “preparing students to answer certain standard types of questions quickly and accurately.” This seemed to contradict the stated philosophy of the College Board (2012b), which stated that AP Calculus is “primarily concerned with developing the students’ understandings of the concepts of calculus and providing experience with its methods and applications.” Leaders in the AP Calculus community have attempted to assist teachers in this regard by providing online pacing guides (McMullin, 2011).

In addressing concerns that the Calculus AB syllabus is “very extensive,” Bressoud (2010a) noted that the syllabus must “include every topic that is commonly covered in the first semester of college calculus.” In order to determine what topics will be included in the syllabus, the 300 mathematics departments that receive the most AP Calculus examination scores are surveyed, and any topics ranked “important” by a “substantial” percent of the respondents are included (although it is not clearly stated what percentage constitutes “substantial”). It also appears that at some point in the past, a seemingly random survey of colleges and universities was also conducted (College Board, 2004). It is clear in the results that only 169 institutions participated in the survey, and several were smaller schools (including community colleges). It has also been reported that only 30-40 percent of schools typically responded to this survey (Gollub et al., 2002), provoking concerns over whether the survey results are consistent with actual university syllabi. It is unclear if these seemingly random surveys are still conducted, or if the College Board has since amended its survey methods to only sample the aforementioned 300 mathematics departments.

Still, there have been recommendations to add content to the AP calculus curriculum—if not officially by the College Board, then individually by teachers on a case-by-case basis. Gollub et al. (2002) noted that the AP Calculus curriculum includes real-world problems, but only those “that are most manageable,” with “relatively few applications requiring deeper investigation.” It was later recommended that teachers add more modeling activities and opportunities to investigate proofs, suggesting that teachers “challenge their students with problems that may go beyond the scope of the syllabi, teaching a course that is more demanding than the tests.”

Gollub et al. (2002) also noted that there is “considerable anecdotal evidence” that by encouraging students to take AP Calculus courses in high school, some students are “rushed through prerequisite courses without learning the material well enough." Evidence of this trend was shown in two different studies. In the first study (U.S. Department of Education, 2008, as cited in Bressoud, 2012), 31% of students from the high school class of 1992 who completed a calculus course during their senior year had to enroll in a precalculus course once they got to college. In the second study (National Science Board, 2010, as cited in Bressoud, 2012), 17% of students from the high school class of 2004 who had taken a high school calculus course had to take remedial mathematics once they got to college. Therefore, in an attempt to curb this trend, joint statements of the Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM) from both 1986 and 2012 stated that students enrolling in calculus while in high school should have already shown "mastery of algebra, geometry, trigonometry, and coordinate geometry" (Gollub et al., 2002; MAA, 2012).

Technical Methods

Instrumentation

As explained by Boudah (2011), a literature review can be used to "provide a rationale/background for study." A review of the literature was used to examine the goals and methods advocated by the College Board in its implementation of the AP Calculus program, as well as recent critiques and criticisms of the AP Calculus program. Based on the findings of the literature review, the focus of the rest of the study was on examining possible shortcomings of the AP Calculus program and investigating ways to help AP Calculus teachers prepare their students to overcome those shortcomings as they transition to undergraduate mathematics studies.

Stratified sampling, defined by Chromy (2006) as "a technique that partitions the sampling frame into collections of sampling units … and selects an independent sample within each sampling stratum," was used to select institutions to examine further. Online perusal of course and department webpages was utilized to find information regarding each institution's Calculus I course, including course descriptions, syllabi, and sample or recent final examinations. Differences between the content of the AP Calculus curriculum and those of college institutions were emphasized throughout this study.

Participants

A variety of undergraduate institutions was selected for this study. The first group of institutions consisted of all "Group I" universities, as determined by the American Mathematical Society (2011). The AMS report listed mathematics departments that award doctorate degrees, with these institutions tiered by group according to the "scholarly quality of program faculty" from highest to lowest. The universities in Group I were those with the highest scores (at least

3.00 points out of 5.00). The Graduate Center, The City University of New York was removed from consideration because its mathematics department awards only post-baccalaureate degrees. Institutions that did not have an adequate amount of data available online for public viewing were also removed from consideration.

To supplement the data from the 29 Group I universities that had adequate information available online, additional data was collected from a stratified sample of Group II and Group III universities. Institutions in Groups II and III also award doctorate degrees, but their scores were lower than those in Group I. In order to limit the scope of this study, the stratified sampling was designed so that ten universities each from Group II and Group III were included, totaling 49 institutions. (Again, those institutions that did not have adequate data available were skipped, and sampling continued with the next randomly selected institution.) Because the University of Hawaiʻi at Mānoa (a Group II institution) is the only program in Hawaiʻi that awards a doctorate in mathematics, it was also included in the study, bringing the number of institutions examined up to a grand total of 50.

Procedure

Information about each university's Calculus I curriculum were found by browsing each institution's website; most of the universities had mathematics department websites that included course descriptions, syllabi, and sample or recent final examinations. Some universities used course websites such as Moodle or Blackboard, which required student login. In these situations, the institutions were not included in further discussion of the collected data; only publicly provided information was utilized in this study. No efforts were made to contact officials at these institutions for assistance.

The resources collected from each institution were evaluated according to the list of course content topics utilized in the *AP Calculus Curriculum Survey* (College Board, 2004), with additional topics added as needed. Topics that are fairly common and are covered relatively well by the AP Calculus curriculum were ignored; only those topics that are not included or covered briefly were utilized in the analysis. For each topic, an institution was listed as including that topic in their curriculum if it was specifically mentioned in a course description, course syllabus, sample final examination, or one of the recent final examinations. Because the list of topics is quite detailed and extensive, those institutions that only provided broad, generic course descriptions were not included in the study.

Based on the data collected, this study examined how the Calculus AB curriculum is different from comparable university courses. A focus was placed upon topics that were included in the curricula of at least 10% of the institutions surveyed. A focus was also placed upon institutions that established their calculus curriculum in particularly unique ways—that is, not the traditional college calculus curriculum. Throughout this study, there was an additional focus on advice and suggested practices to help AP Calculus teachers assist their students in transitioning from AP Calculus to college calculus courses.

In addition to information about the content covered in each course, a list of college calculus textbooks was also compiled. Based on the results collected from the 50 institutions, a list of eight college calculus textbooks was generated. When necessary, some of the calculus topics were evaluated with insights into how the choice of textbook may affect curriculum decisions at the college level. Many universities used similar versions of textbooks written by the same authors—that is, there were some textbooks with different editions, and some that covered topics in different orders (such as early transcendental functions vs. late transcendental functions). In such cases, one encompassing textbook was chosen to represent the family of textbooks. Approximately 77% of surveyed Calculus I courses used a text from Stewart (43%), Hughes-Hallett et al. (19%), Thomas, Weir, & Hass (9%), or Rogawski (6%), which were four of the eight textbooks used in this study (Bressoud, 2011b). Also when needed, information from *Calculus* (8th ed.) by Larson, Hostetler, & Edwards (2006) was used; this text was chosen to represent the typical textbook used in the AP Calculus classroom.

Limitations/Delimitations

Using only Group I, Group II, and Group III institutions eliminated the opportunity to examine colleges and universities that offer only bachelor's and/or master's degrees in mathematics. Of the institutions listed in the AMS study (2011), over 70 percent offered a bachelor's degree as its highest degree in mathematics. It should be noted, however, that the more prestigious universities are expected to have larger mathematics departments and service more students than those in the smaller institutions. As the College Board surveys the universities that receive the most AP examination scores, it is important to see if these institutions' philosophies differ from that of the College Board (Bressoud, 2010a).

Another limitation to this study was the smaller sample size. While there were 47 Group I universities, only 29 of them had adequate information that could be utilized in the study. It would have been ideal to also investigate all 136 of the Group II and Group III institutions, but the scope of that research would have been too great to have been done in a timely manner. Therefore, the decision was made to simply supplement the Group I research with data from Group II and Group III institutions. The sample size of 50 institutions was only one-sixth that of the survey that the College Board regularly holds, and was less than one-third of that in the 2004 survey conducted by the College Board.

The study may have also been limited by the nature of collecting data. Every effort was made to collect data from institutions that provided a detailed course description and/or syllabus, with supplementation from sample or recent final examinations. These sources of information, however, may still have lacked enough information when charted using the topics from the College Board survey (2004); that is, some institutions may actually have included a given topic in their curriculum, but it may not have been specifically stated in the course description/syllabus nor covered in sample or recent final examinations. As a result, some of the percentages may have been lower than expected, as topics were undercounted.

Another limitation of using course descriptions and/or syllabi was the nature of the information posted. Some institutions provided information about the textbook that the course used, along with sections from the text that were covered in the term. These sources sometimes contained descriptions of the sections, or tables of contents for those textbooks were found online that gave the titles of the sections from the textbook. These descriptions and titles, however, may not have stated in detail the subtopics of the section. For example, a section entitled “Limits” may not have specified whether the text covered the Squeeze Theorem, or a section entitled “Advanced Integration Techniques” may not have specified whether those techniques included antidifferentiation by parts, simple partial fractions, or trigonometric substitution. These concerns were addressed by looking for such anticipated topics in final examinations. There is still the possibility, however, that those institutions who teach those topics were not counted in the study, because they did not include that topic on the available final examinations, or did they not have final examinations available online for public examination. Thus, this may have led to some percentages being lower than anticipated due to undercounting.

Results and Analysis: Topics of Note

Antidifferentiation by Parts

Antidifferentiation by parts (or integration by parts) is a topic that is included in the Calculus BC curriculum. It is often included in units on advanced techniques of integration. As stated by Larson et al. (2006):

> If u and v are functions of x and have continuous derivatives, then
>
> $$\int u\,dv = uv - \int v\,du\ .$$

Antidifferentiation by parts was included in the curricula of seven out of 50 institutions (14.0%), and was seen on four out of 24 final examinations (16.7%). This is similar to the College Board's survey results (2004), which stated that 11.2% of 169 institutions surveyed included the topic in their Calculus I curriculum.

Most textbooks include antidifferentiation by parts in a later chapter on advanced integration techniques, bundled with other topics such as antidifferentiation by simple partial fractions and antidifferentiation using trigonometric substitution. This was true for all eight college textbooks examined in this study; Hughes-Hallett et al. (2009), however, also included antidifferentiation by substitution in its late chapter on integration. Since antidifferentiation by substitution is a topic that is commonly included in Calculus I curricula, those using this textbook may have included additional topics such as antidifferentiation by parts.

Covering antidifferentiation by parts in the Calculus AB classroom has minimal benefit with respect to the Calculus AB examination. Definite integral problems that might otherwise require antidifferentiation by parts would need to be assigned as questions on the portions of the examination that require the use of a graphing calculator, which students could use to numerically estimate the value of the definite integral. While antidifferentiation by parts could be used to solve such problems, this would not be recommended—if a student were to make a mistake midway through the calculations, this would lead to a loss of points. Also, there are no potential bonus points for using techniques that are more advanced; a correct value to a definite integral using antidifferentiation by parts would be worth the same as the answer determined using a graphing calculator.

Still, teachers may choose to include this topic in their curriculum. Antidifferentiation by parts is largely procedural, and problems that require repeated use of the technique are often easily solved using the tabular method. While the derivatives are typically not that difficult, antidifferentiation by parts problems sometimes provide additional practice using antidifferentiation by simple substitution; thus, some teachers may decide to pair the lessons together (as Hughes-Hallett et al. did).

Antidifferentiation by Simple Partial Fractions

Antidifferentiation by simple partial fractions is a topic that is covered in the Calculus BC curriculum, dealing with how to find antiderivatives of rational functions that may be too complicated for antidifferentiation by simple substitution. As stated by Larson et al. (2006):

1. **Divide if improper**: If $N(x)/D(x)$ is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator**: Completely factor the denominator into factors of the form

$$(px+q)^m \quad \text{and} \quad (ax^2+bx+c)^n$$

where ax^2+bx+c is irreducible.

3. **Linear factors**: For each factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of *m* fractions.

$$\frac{A_1}{(px+q)}+\frac{A_2}{(px+q)^2}+\cdots+\frac{A_m}{(px+q)^m}$$

4. **Quadratic factors**: For each factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum of *n* fractions.

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

Antidifferentiation by simple partial fractions is often covered in the same unit as other advanced integration techniques, such as antidifferentiation by parts and trigonometric substitution. This topic was included in the curricula of six out of 50 institutions (12.0%), with two out of 24 (8.3%) institutions including such problems on a final examination. These numbers are slightly higher than those from the College Board’s survey (2004), where 5.9% of 169 institutions said they covered antidifferentiation by simple partial fractions in their Calculus I course.

One key note is that the College Board’s course description clearly states that only nonrepeating linear factors should be included in the denominator of the rational function to be decomposed. Therefore, students who do learn antidifferentiation by simple partial fractions in the AP Calculus classroom may not see more complicated problems such as those with terms like $(x-3)^2$ or (x^2+x-5). Colleges and universities, however, may not share in this philosophical restriction. Therefore, teachers of Calculus BC may wish to consider teaching partial fraction decomposition without the restrictions stated by the College Board. Teachers of

Calculus AB who wish to include this topic may want to consider keeping this restriction, though; it is not pressing that Calculus AB students learn the topic in its entirety, and those students who continue on to Calculus BC or a college Calculus II course in the subsequent year (when antidifferentiation by simple partial fractions is typically taught) could then receive instruction in the full topic without restriction.

A key factor for those high school teachers who decide to include antidifferentiation by simple partial fractions in their Calculus AB curriculum may be whether students have had prior exposure to partial fraction decomposition in previous courses. Larson et al. (2005) included partial fraction decomposition in their precalculus text's section on multivariable linear systems, going so far as to foreshadow, "One of the most important applications of partial fractions is in calculus." For students who have had previous exposure to partial fraction decomposition, it may be much easier to pick up on antidifferentiation by simple partial fractions, as much of the integration is based on the antiderivatives of $\frac{1}{u}$ and $\frac{1}{u^2}$, which are commonly seen in antidifferentiation problems.

The number of institutions including antidifferentiation by simple partial fractions is still small, so it is not necessary to include this topic in the Calculus AB curriculum at this time. Noting that the 2012 sample results show a higher percentage than the results from the 2004 sample, however, it may be possible that more schools are looking at adding this topic to their Calculus I curriculum. It is probably too soon to assume that this change will be occurring shortly, as the evidence is hardly overwhelming; at the same time, however, AP Calculus teachers may want to follow the possibility of such a trend. If this continues, Calculus AB teachers may wish to introduce this topic into their curriculum voluntarily, or to work with their precalculus teachers to introduce partial fraction decomposition in that course's curriculum (if they have not yet done so).

Antidifferentiation Using Trigonometric Substitution

Antidifferentiation using trigonometric substitution is a topic that is not covered in the AP Calculus curriculum. As stated by Larson et al. (2006), for $a > 0$:

1. For integrals involving $\sqrt{a^2 - u^2}$, let $u = a \cdot \sin(\theta)$.
   Then $\sqrt{a^2 - u^2} = a \cdot \cos(\theta)$, where $-\pi/2 \leq \theta \leq \pi/2$.

2. For integrals involving $\sqrt{a^2 + u^2}$, let $u = a \cdot \tan(\theta)$.
   Then $\sqrt{a^2 + u^2} = a \cdot \sec(\theta)$, where $-\pi/2 < \theta < \pi/2$.

3. For integrals involving $\sqrt{u^2 - a^2}$, let $u = a \cdot \sec(\theta)$.
   Then $\sqrt{u^2 - a^2} = \pm a \cdot \tan(\theta)$, where $0 \leq \theta < \pi/2$ or $\pi/2 < \theta \leq \pi$.
   Use the positive value if $u > a$ and the negative value if $u < -a$.

Antidifferentiation using trigonometric substitution is often covered in the same unit as other advanced integration techniques, such as antidifferentiation by parts and antidifferentiation using simple partial fractions. This topic was included in the curricula of five out of 50 institutions (10.0%), with only one out of 24 (4.2%) institutions including such problems on a final examination. These numbers are comparable to those from the College Board's survey (2004), where 6.5% of 169 institutions said they covered antidifferentiation using trigonometric substitution in their Calculus I course.

In a way, antidifferentiation using trigonometric substitution is just an extension of antidifferentiation by simple substitution, which is included in the Calculus AB curriculum and in a majority of Calculus I curricula. That said, since it is not required for the Calculus AB examination, and since it is covered by so few Calculus I courses, it does not seem very beneficial to do a full-fledged lesson on this topic. Teachers who wish to challenge their students, however, may want to consider supplementing lessons on antidifferentiation by simple substitution through the assigning of problems that explicitly state what the trigonometric substitution should be; thus, students are not burdened to try and figure out the substitution on their own, while still gaining some exposure to the more advanced topic (albeit at a more intermediate level).

Calculus BC teachers, however, may want to take a more serious look at supplementing their curriculum with this topic. While only 6.5% of 169 institutions included antidifferentiation using trigonometric substitution in their Calculus I course, a whopping 86.4% of Calculus II courses covered it (College Board, 2004). Thus, it is a safe assumption that most students who have completed the typical two-semester single variable calculus sequence have seen antidifferentiation using trigonometric substitution. Even though it is not required for either AP Calculus course, it may be prudent for Calculus BC teachers to voluntarily include this topic in their curricula.

Applications of Integrals in Various Contexts to Model Physical, Biological, or Economic Situations

Applications of integrals in various contexts to model physical, biological, or economic situations is a topic that is not generally included in the AP Calculus curriculum. While the course description set the expectation that "appropriate integrals are used in a variety of applications to model physical, biological, or economic situations," the AP Calculus examination has typically been very specific in terms of what those situations are (College Board, 2012b). Such situations include, but are not limited to, finding lengths of telephone wires (arc length, a Calculus BC topic) and the average density of a rod (average values of functions, a Calculus AB topic). In examining final examinations of various institutions, several had problems dealing with exponential growth and decay that could be solved using separable differential equations, which is also a Calculus AB topic.

As the aforementioned situations are adequately covered by the AP Calculus curriculum, the focus here is on applications that are not specifically included in the AP Calculus curriculum. Such applications were included in the curricula of five out of 50 institutions (10.0%), and were tested by two out of 24 institutions (12.5%). Examples of such applications, as covered by Salas et al. (2007) and Stewart (2012), include fluid pressure and fluid force, hydrostatic pressure and force, consumer surplus, blood flow and Poiseuille's Law, and cardiac output.

Of note was the fact that the College Board's survey (2004) not only included the broad class of "physical, biological, or economic situations," but also specified "applications of integrals to work and force," as well as "applications of integrals to moments, mass, and center of mass" (neither of which are AP Calculus topics). Because textbook authors typically grouped these and other topics into a chapter on applications of integrations, those institutions that cover the more common topics (such as finding areas of regions or finding volumes of solids with known cross sections) may tend to include the other application topics as well.

While some institutions did go into applications of integration, there were a significant number that did not, as noted by Bressoud (2010a). The scope of this study's research did not look into Calculus II courses, so it was not determined what percentage of institutions covers these topics in Calculus II. The College Board's survey in 2004, however, indicated that 84.2% of the 169 institutions surveyed included these applications of integrals in various contexts to model physical, biological, or economic situations in their curricula. Therefore, while it may not be necessary for Calculus AB teachers to include such topics, it may be wise for Calculus BC teachers to investigate supplementing their curriculum with such additional topics. While they may not be tested on the Calculus BC examination, those topics may be useful as project-based learning opportunities.

Applications of Integrals to Work and Force

Applications of integrals to work and force is a topic that is not included in the AP Calculus curriculum. It is often included in units on applications of integration. As stated by Larson et al. (2006):

> If an object is moved along a straight line by a continuously varying force $F(x)$, then the **work** W done by the force as the object is moved from $x = a$ to $x = b$ is
>
> $$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \Delta W_i = \int_a^b F(x)\, dx .$$

This definition is utilized in problems involving Hooke's Law, Newton's Law of Universal Gravitation, and Coulomb's Law. It can also be used to determine the work required to lift an object or pump a liquid, as well as the work performed by an expanding gas (such as with pistons). (Note: This list is not exhaustive of the types of exercises in this topic.)

Applications dealing with work and force were included in the curricula of seven out of 50 institutions (14.0%), with three out of 24 (12.5%) institutions including work and force problems on a final examination. These numbers are actually less than those from the College Board's survey (2004), where 21.9% of 169 institutions said they covered work and force problems in their Calculus I course. It should be noted, however, that 78.7% of the 169 institutions said they covered work and force problems in their Calculus II course; meanwhile, this topic is not included in the AP Calculus curriculum at all.

The AP Calculus course description states that "students should be able to adapt their knowledge and techniques to solve" applications of integral problems such as these, even though work and force problems are not specifically included in the course description, and are not tested on the AP Calculus examination (College Board, 2012b). It should be noted, however, that there are no new antidifferentiation techniques that are required to solve such problems; this is clearly seen by the fact that five of the eight textbooks examined in this study cover work and force problems prior to their chapter on advanced integration techniques (Anton et al., 2009; Briggs & Cochran, 2011; Salas et al., 2007; Smith & Minton, 2007; Thomas et al., 2010). The challenge is simply in setting up the definite integral correctly, so that it accurately represents the scenario from the problem. Of course, this can be a considerable challenge for students, and some may wonder why they are asked to struggle through such problems in the classroom, when they will not be seeing any such problems on the AP Calculus examination.

Because the number of institutions that cover work and force problems in Calculus I is relatively low, it may not be necessary to cover this topic in the Calculus AB classroom. High school physics teachers, however, may be interested in assigning problems such as these to their students. While not every high school offers a calculus-based physics course (such as AP Physics C), it may be possible to teach students in the physics classroom how to set up the integrals for such problems, even though some students may not have the calculus background to compute the integrals by hand. For those students, graphing calculator technology can easily determine the numerical values. Thus, it may still be possible for Calculus AB students to stretch their understanding through work and force problems, assuming most are dual-enrolled in a physics course at the time.

While the primary focus of this study is on Calculus AB, it should be noted that Calculus BC teachers may want to strongly consider supplementing their curriculum with this topic. As previously stated, 78.7% of 169 institutions surveyed included this topic in their Calculus II curriculum (College Board, 2004). Combined with the 21.9% of 169 institutions surveyed that included this topic in their Calculus I curriculum, and assuming that there were some institutions that covered the topic in both courses, this accounts for nearly all of the institutions. Thus, it is safe to assume that many universities will expect that students who have gone through a two-semester single-variable calculus sequence will have seen integration problems dealing with work and force.

Curve Sketching

Analysis of curves is a topic that is universally included in Calculus I curricula, with 95.9% of 169 institutions covering this topic in Calculus I (College Board, 2004). There appears, however, to be a difference in philosophy between the AP Calculus curriculum and traditional college calculus curricula regarding how to assess understanding of this topic. A traditional college way of doing so is to require students to sketch a curve without a graphing calculator, based on the behavior of the function's first and second derivatives (typically denoted through student-created sign charts). Focusing specifically on the subject of curve sketching, 11 out of 24 institutions (45.8%) included this topic on a final examination.

The AP Calculus examination, however, does not require students to perform any curve sketches. Instead, the course description simply requires that students understand "the notions of monotonicity and concavity," as well as "corresponding characteristics of the graphs of f, f', and f''" (College Board, 2012b). A typical AP Calculus examination question might provide the graph of f', and ask students to identify the graph of f, or vice versa. Another typical AP Calculus examination question might provide information about f, f', and f'' in a table, and ask about the locations of relative extrema or points of inflection.

Indeed, these questions do test students on their understanding of the topic, and there were a few universities that asked similar questions on their final examinations. Requiring students to actually sketch a curve, however, can potentially help students in solidifying their understanding. By requiring students to combine the concepts of increasing/decreasing and concavity, curve sketching helps students to grasp visually how the function behaves. While multiple-choice questions on the AP Calculus examination test similar understanding, the students can potentially use good test-taking strategies as a crutch, eliminating answer choices as they determine where something is wrong.

Requiring students to sketch out curves can be a struggle, however, as there are many details that need to be checked when determining the accuracy of student responses. Since there are a number of institutions that include curve sketching in their Calculus I curriculum, teachers are advised to include this in their curriculum. To supplement these units, however, teachers may wish to ask questions that are geared more toward the AP Calculus style of examination, but with some minor tweaks. For multiple-choice problems, for example, students may be required to justify their correct choices in order to receive full credit, so that they cannot just depend on eliminating answer choices when they see something wrong. Furthermore, an answer choice of "none of the above" may help to encourage students to look at the entirety of an answer choice, once they have eliminated all others. On a similar note, another type of question could provide students with information about the behavior of the function as well as a proposed sketch; students would then have to justify if the provided sketch is accurate, or explain where there are errors.

Definite Integral as Limit of Riemann Sums

Representing a definite integral as a limit of Riemann sums is typically a common topic in Calculus I curricula, with 85.2% of 169 institutions including it in their course (College Board, 2004). It was also included in final examinations at seven out of 24 institutions (29.2%). As defined by Larson et al. (2006):

> If *f* is defined on the closed interval $[a, b]$ and the limit $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$ exists, then *f* is **integrable** on $[a, b]$ and the limit is denoted by
>
> $$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx\,.$$
>
> The limit is called the **definite integral** of *f* from *a* to *b*. The number *a* is the **lower limit** of integration, and the number *b* is the **upper limit** of integration.

The seven institutions that included problems dealing with this topic provided the Riemann sums, and asked students to determine the solution. Several of the examinations gave the hint that the students should change the given Riemann sum into a definite integral, implying that it could then be easily solved using the Fundamental Theorem of Calculus.

Some textbooks, such as that by Larson et al. (2006), also cover how to use summations to determine the areas of various regions in the plane. These problems actually require calculating of the summations, not the Fundamental Theorem of Calculus, in order to determine the areas. In perusing the final examinations at the seven institutions, however, none of them appeared to have problems such as these. At those institutions, it appeared that the priority was in recognizing the connection between Riemann sums and definite integrals.

As such, AP Calculus teachers may want to consider striking the actual evaluation of summations from their curriculum, if they had already added it. While it seems very natural to include the topic of evaluating summations while going through the unit on definite integrals, these problems do not appear on the AP Calculus exam, and did not appear on the college final examinations that were studied. The problems constitute a nice review of summations from precalculus, but are ultimately unnecessary for the AP Calculus classroom.

The ε-δ Definition of a Limit

The ε-δ (epsilon-delta) definition of a limit is a topic that is not included at all in the AP Calculus curriculum. Often taught in introductory real analysis courses, it is also taught in some Calculus I courses, as well as in multivariable calculus courses (with the equivalent definition in vector calculus). As stated by Larson et al. (2006):

> Let *f* be a function defined on an open interval containing *c* (except possibly at *c*) and let *L* be a real number. The statement
>
> $$\lim_{x \to c} f(x) = L$$
>
> means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if
>
> $$0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

The ε-δ definition was included in the curricula of 13 out of the 50 institutions (26.0%), and on 3 out of 24 final examinations (12.5%).

Recently, some textbooks have begun devoting separate sections to this formal definition of a limit. These texts include those by Anton et al. (2009), Briggs & Cochran (2011), Rogawski (2012), Salas et al. (2007), Smith & Minton (2007), Stewart (2012), and Thomas et al. (2010). The text by Hughes-Hallett et al. (2009) was the only one of the eight examined that did not have a section solely for the ε-δ definition of a limit. With this option, some universities made an explicit choice to skip over this section in their curricula, while others still chose to keep it.

Since there is absolutely no advantage to learning the ε-δ definition of a limit in preparation for the AP Calculus examination, it is not advised to cover this material until after the AP Calculus examination. If high school teachers decide to teach this, however, one point of emphasis is that students should map out their proof before they begin to write it. Often, the final proofs begin with defining δ as a function dependent on ε. Some students struggle with this, unable to figure out how that relationship magically came to be. By helping students to work backwards from $f(x)-L$ to $x-c$ using various algebraic manipulations, this should help students to see the optimal choice for defining δ.

For example, given the problem $\lim_{x\to 1}(2x+5)$. Noting that this limit will be $L=7$, students can start planning with $f(x)-7=(2x+5)-7=2x-2$; thus, the final conclusion should be $|2x-2|<\varepsilon$. Now they need to manipulate this to get $x-1$; that is, they need to divide both sides of the inequality by 2, to get $|x-1|<\varepsilon/2$, which is in the form $|x-c|<\delta$. Thus, students can start off their proof by setting $\delta=\varepsilon/2$, so that:

> Choose $\varepsilon>0$. Then for each ε, there is a $\delta=\varepsilon/2$, and $\delta>0$. Then $0<|x-1|<\delta$ implies $2(0)<2|x-1|<2(\varepsilon/2)$, or $0<|2x-2|<\varepsilon$. This can be rewritten as $|(2x+5)-7|<\varepsilon$ or $|f(x)-7|<\varepsilon$, proving by the ε-δ definition of a limit that $\lim_{x\to 1}(2x+5)=7$.

Estimation Using Differentials

The AP Calculus curriculum has placed a heavy emphasis on the concept of the derivative. One of the topics that has been of heavy emphasis in previous years is local linearity, in that functions can be approximated by linear functions in sufficiently small neighborhoods. It is along that line that students are often asked on AP Calculus examinations to determine the tangent line to a curve at a given point, and to use that tangent line to approximate a function value near the given point.

One topic that is not required to be taught by the AP Calculus curriculum, however, is the concept of differentials. Students do see something along the lines of differentials when they deal with separable differential equations, but they are not required to learn about differentials in the strictest defined sense. Furthermore, students are not required to learn how to use differentials to approximate function values. From Larson et al. (2006):

> Differentials can be used to approximate function values. To do this for the function given by $y = f(x)$, you use the formula
>
> $$f(x+\Delta x) \approx f(x) + dy = f(x) + f'(x)\,dx$$
>
> which is derived from the approximation $\Delta y = f(x+\Delta x) - f(x) \approx dy$.

Keen observers, however, will note that the approximation $f(x+\Delta x) \approx f(x) + f'(x)\,dx$ shares many parallels with the tangent line method of approximation. If the line through points $(x, f(x))$ and $(x+\Delta x, f(x+\Delta x))$ has approximate slope $f'(x)$, then $\frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x} \approx f'(x)$, or $f(x+\Delta x) \approx f(x) + f'(x) \cdot \Delta x$. Using $\Delta x \approx dx$, this obtains the same formula stated above.

Estimation using differentials was a topic that was included in 36 out of the 50 curricula (72.0%) examined in this study, and was tested on 12 out of 24 final examinations (50.0%). While some institutions specified for approximations using tangent lines (like those on the AP Calculus examination), these institutions noted here specifically required the use of differentials. While there are many similarities in the two methods, there are enough subtle differences to distinguish between the two. Therefore, it should be noted that college students who use the tangent line method to solve a problem asking for a solution with differentials (or vice versa) may be penalized for doing so at some institutions.

Thus, AP Calculus teachers may wish to supplement their curriculum with a lesson on differentials. Commonly used AP Calculus textbooks such as Larson et al. (2006) have a specific section on differentials, as do numerous college calculus textbooks (Anton et al., 2009; Briggs & Cochran, 2011; Stewart, 2012; Thomas et al., 2010). While most problems at the college level dealt with approximations of radicals (such as $\sqrt{9.2}$ or $\sqrt[3]{8.4}$), textbook problems dealing with applications (such as area, volume, and physics) may be useful to help stretch student understanding of the concepts (rather than just symbolic manipulation and insertion of numbers into formulas).

Finding the Length of a Curve

Finding the length of a curve (also known as determining arc length) is a topic that is covered in the Calculus BC curriculum. It is often included in units on applications of integration. As stated by Larson et al. (2006):

> Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of *f* between *a* and *b* is
>
> $$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx .$$

A similar definition holds for a smooth curve given by the equation $x = g(y)$ on the interval $[c, d]$. This topic was included in the curricula of seven out of 50 institutions (14.0%), with three out of 24 (12.5%) institutions including such problems on a final examination. These numbers are slightly less than those from the College Board's survey (2004), where 17.2% of 169 institutions said they covered arc length problems in their Calculus I course.

Problems where the function is defined as a polynomial seem relatively simple; the most difficult part of such problems may be the algebra required to square the derivative term, which could possibly consist of three or more terms. Where more complicated problems abound, however, is when the models include non-polynomial functions. In fact, Larson et al. (2006) even included an example and several exercises that utilized hyperbolic functions such as $\cosh(u)$, when hyperbolic functions are not included in the AP Calculus curriculum. Covering arc length late in textbooks is not a universal concept, however. Anton et al. (2009), Briggs & Cochran (2011), Rogawski (2012), Smith & Minton (2007), and Thomas et al. (2010) all covered arc length prior to their chapters on advanced integration techniques, some of which include hyperbolic functions. And in classrooms where hyperbolic functions are not covered, those problems dealing with hyperbolic functions can always be excluded from the assigned work.

Since arc length is strictly a Calculus BC topic, there is no benefit to learning this topic for the Calculus AB examination, other than obtaining extra practice working with integrals. As such, it does seem unnecessary to cover this topic, especially since the number of institutions that cover this topic in Calculus I is still relatively low. For those teachers who do cover this topic in Calculus AB, however, it should be noted that the Calculus BC curriculum also includes arc lengths of curves defined parametrically. Assuming those Calculus AB courses do not cover parametric curves also, it is important to communicate to Calculus BC teachers at those high schools that students who took Calculus AB only learned how to determine arc lengths of curves expressed non-parametrically.

Finding Volumes of Solids Using the Shell Method

Finding volumes of solids using the shell method is a topic that is not included at all in the AP Calculus curriculum, as all volume of revolution problems can be done using disks or washers (Levine, 2011; McMullin, 2011 April 14). It is an alternate technique used to determine volumes of revolution, essentially using cylindrical shells of infinitesimal thickness; thus, the integrand consists of the lateral area ($2\pi rh$) of any given shell (the functions r and h dependent on x for a vertical axis of revolution, or dependent on y for a horizontal axis of revolution), with the differential term representing the thickness. The shell method was included in the curricula of 16 out of the 50 institutions (32.0%), but only on three out of 24 final examinations (12.5%).

The lower number of institutions that include the shell method in Calculus I may be less than other topics because of two reasons. The first is that some institutions do not include applications of integrals in their Calculus I curriculum. As stated by Bressoud (2010a), some institutions assume that students taking Calculus I will continue on to take Calculus II, so they only teach up to an introduction to integration (often up to integration by substitution, according to the survey of course descriptions and syllabi). They then continue with applications of integration (including volumes of revolution) in Calculus II. For those institutions that do include applications of integrals in Calculus I, they typically include volumes of revolution using disks and washers, but leave out the shell method. Seven of the eight textbooks investigated in this study have separate sections for volumes with disks/washers and volumes with shells, and many course descriptions explicitly left out the section covering the shell method (Anton et al., 2009; Briggs & Cochran, 2011; Rogawski, 2012; Salas et al., 2007; Smith & Minton, 2007; Stewart, 2012; Thomas et al., 2010). Only Hughes-Hallett et al. (2009) had a solo section devoted to volumes together, entitled “Areas and Volumes.”

The shell method is a legitimate calculus technique, so it can be utilized on the AP Calculus examination (Foerster, 2012). McMullin (2011 April 14) recommended, however, that this be taught in the time after the examination, if the schedule so allows. Because volumes of revolution is typically one of the last topics covered in the curriculum, teachers may indeed wish to defer covering the shell method until after the AP Calculus examination, if they cover it at all. The time spent covering the shell method might otherwise best be used as review sessions for the AP Calculus examination. Students who receive advanced placement through their AP examination results are advised to check with their institution to see whether they should learn the shell method independently prior to entering Calculus II.

Hyperbolic Functions

Hyperbolic functions is a topic that is not covered at all in the AP Calculus curriculum. This class of exponential functions shares many similarities to the class of trigonometric functions. As stated by Larson et al. (2006), the first two hyperbolic functions are defined as $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$. The other functions-- $\tanh(x)$, $\coth(x)$, $\operatorname{sech}(x)$, and $\operatorname{csch}(x)$ --are defined like their trigonometric counterparts in terms of $\sinh(x)$ and $\cosh(x)$. In addition to some hyperbolic identities that share some similarities with the trigonometric identities, there are also the derivatives and antiderivatives:

$\frac{d}{dx}[\sinh(u)] = \cosh(u) \cdot u'$	$\int \cosh(u)\, du = \sinh(u) + C$
$\frac{d}{dx}[\cosh(u)] = \sinh(u) \cdot u'$	$\int \sinh(u)\, du = \cosh(u) + C$
$\frac{d}{dx}[\tanh(u)] = \operatorname{sech}^2(u) \cdot u'$	$\int \operatorname{sech}^2(u)\, du = \tanh(u) + C$
$\frac{d}{dx}[\coth(u)] = -\operatorname{csch}^2(u) \cdot u'$	$\int \operatorname{csch}^2(u)\, du = -\coth(u) + C$
$\frac{d}{dx}[\operatorname{sech}(u)] = -\operatorname{sech}(u)\tanh(u) \cdot u'$	$\int \operatorname{sech}(u)\tanh(u)\, du = -\operatorname{sech}(u) + C$
$\frac{d}{dx}[\operatorname{csch}(u)] = -\operatorname{csch}(u)\coth(u) \cdot u'$	$\int \operatorname{csch}(u)\coth(u)\, du = -\operatorname{csch}(u) + C$

Hyperbolic functions were included in the curricula of six out of 50 institutions (12.0%), but none of the 24 institutions that had final examinations available included anything with hyperbolic functions. The College Board's survey (2004) indicated 10.7% of 169 institutions said they covered hyperbolic functions in their Calculus I course.

The eight textbooks included in this study covered hyperbolic functions in different ways. Anton et al. (2009) and Hughes-Hallett et al. (2009) included the topic in their chapters on applications of integrals. Briggs & Cochran (2011), however, covered hyperbolic functions in their unit on differentiation, prior to even beginning integration. Rogawski (2012), Salas et al. (2007), Stewart (2012), and Thomas et al. (2010) all included hyperbolic functions in their chapters on derivatives and antiderivatives of transcendental functions. Smith & Minton (2007) covered hyperbolic functions in an introductory review chapter, but it was not clear from the table of contents when they cover the derivatives and antiderivatives of hyperbolic functions; it is assumed that the derivatives were covered early, as there is no chapter dedicated solely to transcendental functions like the four mentioned earlier.

Much like was done at some of the college institutions, it is possible to simply skip any sections on hyperbolic functions; thus, even Calculus AB teachers with textbooks that cover hyperbolic functions around the same time as other Calculus AB topics can easily cut out those sections, as well. While there are some similarities to the derivatives, antiderivatives, and identities of trigonometric functions, there are enough differences that may cause confusion for students who are simply trying to get the trigonometric rules straight. As there is no discernible reason to introduce hyperbolic functions in the Calculus AB curriculum, it is recommended that teachers not voluntarily add it to their classrooms. That said, those teachers who do are strongly recommended to emphasize the differences between the trigonometric functions and the hyperbolic functions, particularly when dealing with positive and negative signs.

Calculus BC teachers, however, may want to take a more serious look at supplementing their curriculum with this topic. While only 10.7% of 169 institutions included hyperbolic functions in their Calculus I course, this number grew to 50.9% when examining Calculus II courses (College Board, 2004). While considering that some institutions may have covered derivatives of hyperbolic functions in Calculus I and antiderivatives of hyperbolic functions in Calculus II (and thus are being double-counted), it is still safe to say that at least half of the 169 institutions covered hyperbolic functions at some point in time during the two-semester single-variable calculus sequence. Thus, the Calculus BC teacher may want to investigate the mathematics departments of those universities that their high school students tend to matriculate to; if those institutions include hyperbolic functions in their curriculum, the Calculus BC teacher may want to introduce it at some point.

L'Hôpital's Rule

L'Hôpital's Rule (also spelled l'Hospital's Rule) is a topic that is included in the Calculus BC curriculum. The rule applies to limits, but requires knowledge of differentiation before it can be utilized. As stated by Larson et al. (2006):

> Let *f* and *g* be functions that are differentiable on an open interval (a, b) containing *c*, except possibly at *c* itself. Assume that $g'(x) \neq 0$ for all *x* in (a, b), except possibly at *c* itself. If the limit of $f(x)/g(x)$ as *x* approaches *c* produces the indeterminate form $0/0$, then
>
> $$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$
>
> provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as *x* approaches *c* produces any of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

L'Hôpital's Rule was included in the curricula of 39 out of 50 institutions (78.0%), by far the greatest total of all the topics that are not included in the Calculus AB curriculum. Of the 24 institutions that provided final examinations, 19 of them (79.2%) included limit problems that required l'Hôpital's Rule in order to be solved correctly. This contrasts greatly from the College Board's survey results (2004), which stated that 49.1% of 169 institutions surveyed included the topic in their Calculus I curriculum.

For some institutions, including l'Hôpital's Rule in their curricula may have to do with their choice of textbook. Anton et al. (2009), Briggs & Cochran (2011), Hughes-Hallett et al. (2009), Rogawski (2012), and Smith & Minton (2007) included a section on l'Hôpital's Rule prior to covering integration. Stewart (2012) and Thomas et al. (2010) covered derivatives and antiderivatives of transcendental functions late in their textbooks; they included l'Hôpital's Rule in the same chapter as well. Because most Calculus I curricula include the derivatives and antiderivatives of these transcendental functions, institutions using these texts may choose to teach l'Hôpital's Rule at this time as well.

Whether the text covers transcendental functions early (prior to an introduction to integration) or late (after covering basic integration), it appears that l'Hôpital's Rule can be covered, at least at an introductory level, in the Calculus AB classroom. Because it is a Calculus BC topic, there are no questions on the Calculus AB examination that require the use of l'Hôpital's Rule in order to be solved. This does not mean, however, that l'Hôpital's Rule cannot be used; methods are not checked on multiple-choice questions, and for free-response questions, "readers will accept any mathematically correct solution to a problem as long as it involves calculus" (Foerster, 2012).

Therefore, it is recommended that l'Hôpital's Rule be introduced to students in Calculus AB if time allows. Problems involving the indeterminate forms $0 \cdot \infty$, 1^{∞}, ∞^{0}, 0^{0}, and $\infty - \infty$ can probably be ignored or introduced briefly when taught in the Calculus AB classroom, because no limit problems on the Calculus AB examination should lead to such an intermediate result. Instead, techniques for transforming these indeterminate forms so that l'Hôpital's Rule can be used are best saved for the Calculus BC classroom or for after the AP Calculus examination (as some schools have as much as a month after the AP Calculus examination before the end of the term). Students who learn the basics of l'Hôpital's Rule and who receive advanced credit from the Calculus AB examination are also advised to consult with the mathematics department at their respective universities to investigate whether they need to learn the advanced l'Hôpital's Rule problems on their own. It should also be noted that l'Hôpital's Rule will probably be readdressed when learning about improper integrals and convergence of sequences, which are both Calculus BC/Calculus II topics.

Logarithmic Differentiation

Techniques for differentiation are universally covered in Calculus I courses, and much of those techniques are included in the AP Calculus curriculum. While the AP Calculus course description specifies that students should know the derivatives of logarithmic functions (College Board, 2004), it does not specify that students need to understand and utilize the technique of logarithmic differentiation—that is, taking the logarithm of both sides of an equation $y = f(x)$ and deriving implicitly to obtain dy/dx. This was a topic that was included on final examinations at nine out of 24 institutions (37.5%).

It is unclear when colleges cover logarithmic differentiation in the classroom; of the eight textbooks examined in this study, none of them mentioned logarithmic differentiation in a table of contents. Thus, it is unclear whether those texts actually included logarithmic differentiation or not; it is reasonable to assume, though, that they may include the technique somewhere in the text, probably in one of the sections on derivatives of logarithmic functions. In five of the eight textbooks (Anton et al., 2009; Briggs & Cochran, 2011; Salas et al., 2007; Stewart, 2012; Thomas et al., 2010), implicit differentiation is covered prior to derivatives of logarithmic functions, so it would be possible to include logarithmic differentiation in the discussion of derivatives of logarithmic functions. This also holds true for the text from Larson et al. (2006), which was confirmed to include logarithmic differentiation as a technique in its section on the derivative of the natural logarithmic function.

While logarithmic differentiation is not tested on the AP Calculus examination, covering this technique may still be beneficial to the students. For those classes that use a textbook with early coverage of transcendental functions, covering logarithmic differentiation can provide extra practice to reinforce the new skills that students are learning at the time. For those classes that use a textbook with late coverage of transcendental functions, covering logarithmic differentiation can provide a nice review of implicit differentiation, since those texts generally have a gap in coverage that is filled with an introduction to integration. Either way, covering logarithmic differentiation can also provide a nice review of the properties of logarithms, which can help students understand how previously acquired knowledge can make complicated derivatives like $\frac{d}{dx}\left[\frac{(x-2)^2}{\sqrt{x^2+1}}\right]$ much simpler. Exposure to such techniques may possibly help students in the development of their mathematical understanding and perseverance.

Modeling Rates of Change

Modeling rates of change (also known as related rates) is a topic that is universally included in Calculus I curricula, with 92.3% of 169 institutions covering it in Calculus I (College Board, 2004). Modeling rates of change was so prevalent a topic that when conducting research of the 50 institutions, it was deemed unnecessary to note what percentage included it in their curriculum, knowing that it would be a very significant amount. Of greater importance, however, was the percentage of institutions that included modeling rates of change problems on their final examinations; that turned out to be 18 out of 24 institutions (75.0%).

Modeling rates of change problems are clearly included in the AP Calculus curriculum; there are, however, concerns about the level of rigor with which those questions are asked on the

AP Calculus examination. In examining Larson et al.'s (2006) section on related rates, 17 out of the 54 exercises were word problems that did not explicitly state an equation that needed to be used, showing only a diagram instead. With those problems, students are expected to develop their own model and use it to solve the related rates problem. The problems included on college examinations were similar in nature to these problems.

Meanwhile, in examining the last ten years of AP Calculus free-response questions (including Form B examinations), six problems dealt with related rates in some way; of those six, only four required students to set up some sort of model, while the other two explicitly stated the equation that was to be used (College Board, 2012a; McMullin, 2012). Those four questions, however, had students model in a way that was different from those problems by Larson et al. With the four AP Calculus examination problems, students were typically given scenarios where a liquid was being pumped into a container at a given rate, say $A(t)$, while a subsequent pump was removing liquid from the container at a given rate, say $B(t)$. Thus, questions could be as simple as asking whether the volume of the container was increasing or decreasing at a given time, which would be indicated by the sign of $A(t) - B(t)$. While these problems still model rates of change, they do not do so in the manner that the textbooks typically cover.

While not explicitly stated by any members of the AP Calculus community, it can be theorized that related rates problems on the AP Calculus examination have taken this less-than-traditional approach in order to minimize the risk that student error brings to the problem. By removing the requirement for students to generate their own complicated models, the AP Calculus examination can concentrate more on examining calculus understanding. While there may be some virtue to this argument, the fact remains that colleges are still asking related rates problems in the traditional way, where students need to generate their own equations to represent physical relationships. Thus, AP Calculus teachers should be sure to introduce traditional related rates questions to their students, even if the trend is moving away from such questions on the AP Calculus examination.

Newton's Method

Newton's Method, also known as the Newton-Raphson Method, is a topic that is not included at all in the AP Calculus curriculum. It was, however, previously included in the curriculum prior to 1998 (McMullin, 2012 January 8). It is a technique that utilizes tangent lines to approximate the zeros of a function. As stated by Larson et al. (2006):

> Let $f(c) = 0$, where *f* is differentiable on an open interval containing *c*. Then, to approximate *c*, use the following steps.
>
> 1. Make an initial estimate x_1 that is close to *c*. (A graph is helpful.)
> 2. Determine a new approximation $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.
> 3. If $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.
>
> Each successive application of this procedure is called an iteration.

Newton's Method was included in the curricula of 20 out of the 50 institutions (40.0%), and on eight out of 24 final examinations (33.3%). It is often taught during the chapter on applications of differentiation (Anton et al., 2009; Rogawski, 2012; Salas et al., 2007; Smith & Minton, 2007; Stewart, 2012; Thomas et al., 2010), although Hughes-Hallett et al. (2009) included the topic in an appendix and as part of a project.

It is assumed that Newton's Method was removed from the AP Calculus curriculum because the calculations essentially require the use of a calculator. Scientific calculators are not allowed on the AP Calculus examination (College Board, 2012b), and requiring Newton's Method to determine the zeros of a function while allowing students to use a graphing calculator makes little sense, since the purpose of Newton's Method is to approximate zeros, which the graphing calculators can easily do. Newton's Method could be used during free-response questions, but recent examinations have shown that the examination writers tend to ask about linearization in a simpler manner, asking for a single tangent line without multiple iterations. Doing so also allows the examination writers to test for understanding during the non-calculator portion of the examination, if they so desire.

Linearization is a focus of the AP Calculus curriculum, so AP Calculus teachers may still wish to supplement the curriculum with Newton's Method if they so desire. It is recommended, however, that the theory of Newton's Method be emphasized over the process. Examining Step 2 of the procedure by Larson et al. (2006), it is not quite clear that this equation was derived using the tangent line at $(x_n, f(x_n))$, noting that the line goes through $(x_{n+1}, 0)$. Rather than introducing the procedure, teachers may wish to allow the students to discover the process for themselves through some sort of project, possibly that from Hughes-Hallett et al. (2009).

Optimization

Optimization is a topic that is universally included in Calculus I curricula, with 97.0% of 169 institutions covering it in Calculus I (College Board, 2004). It was so prevalent that when conducting research of the 50 institutions, it was deemed unnecessary to note what percentage included it in their curriculum, knowing that it would be a very significant amount. Of greater importance, however, was the percentage of institutions that included optimization problems on their final examinations; that turned out to be 21 out of 24 institutions (87.5%).

Optimization problems are clearly included in the AP Calculus curriculum, but of note is that the AP Calculus examination now tests students "in ways other than the traditional build the best box type problem" (McMullin, 2011 December 19). The concern is that if a student were to make an error in setting up the model for the optimization problem, then the student would potentially lose the points on subsequent work, even if the calculus were correct. Many of the problems from the college final examinations, however, were of that "build the best box" type of problem; other problems dealt with a function and a given point not on the function, with the problem asking for the point on the function closest (i.e., minimum distance) to the given point. Either way, students had to generate their own mathematical model, and then use that model to solve the optimization problem.

Because the AP Calculus examination does not have the more challenging problems that require students to develop their own models, some teachers may have opted to skip or skim over these problems in favor of those that have the models already set up for students. While these problems would still allow for students to develop the calculus skills required, they lack the rigor that the college examinations require. In essence, providing the models make these optimization problems no more than elevated maximum and minimum problems similar to those asking for the relative extrema of $f(x) = 3x^2 - 5x + 1$, except with added domain and range restrictions that are inherent to the application aspect of the model.

Most, if not all, textbooks require students to generate their own models when working on optimization problems. Most, if not all, college final examinations also require students to generate their own optimization models. It is only on the AP Calculus examination where students are not required to do so. Considering that is only one examination, it is strongly recommended that AP Calculus teachers still introduce their lessons on optimization with the student-created model aspect still included. While teachers may choose to provide some assessment problems with models already generated, there should still be a decent focus on student-generated models.

Precalculus Topics

Out of the 50 institutions investigated, 35 of them (70.0%) included some sort of precalculus material at the beginning of the course, typically taking the first week to review key topics. Furthermore, five of the 24 institutions (20.8%) even included precalculus topics on their final examinations. Such topics typically included material on conic sections, including determining equations of ellipses and hyperbolas. As stated earlier, however, the AP Calculus course description states that "all students should complete four years of secondary mathematics … [with] courses in which they study algebra, geometry, trigonometry, analytic geometry, and elementary functions" (College Board, 2012b). Furthermore, key AP Calculus community leaders have recommended "NOT teaching the first chapter (the review chapter) in any book" (McMullin, 2012 May 22).

With these recommendations, some AP Calculus teachers have decided to give out summer assignments as homework. The summer homework typically covers a review of precalculus topics, as well as limits (if the high school includes limits in the precalculus curriculum). Andres (2012) noted that some students appreciate summer homework, as it helps get their "math brains turned on" prior to the start of the school year. At Moanalua High School, students are assigned summer homework but are not allowed to start on it until July 1; with school starting in late July/early August, students who complete their assignments too early may not retain the knowledge and skills entering the start of the school year (Nagaoka, 2012). Recently, however, other AP Calculus teachers have criticized this summer assignment movement, stating that they choose not to assign any summer homework (Romain, 2012; Snitz, 2012; Swatek, 2012).

Ultimately, the decision whether or not to assign summer homework is up to each individual teacher's prerogative. The College Board does not appear to officially advocate for either side, just that students need to enter the course with a solid enough background. For some schools, summer homework may not be necessary if students must undergo an extensive screening process in order to enroll in AP Calculus; at other high schools with more open enrollment policies, summer homework or an entrance examination may be necessary to ensure that students are truly prepared for the coursework.

It is yet to be determined, however, what options are available for students that have taken and passed the prerequisite courses, yet are deemed unprepared by performance on summer homework or entrance examinations. As previously stated, 31% of students from the high school class of 1992 who completed a calculus course during their senior year had to enroll in a precalculus course once they got to college (U.S. Department of Education, 2008, as cited in Bressoud, 2012). Furthermore, 17% of students from the high school class of 2004 who had taken a high school calculus course had to take remedial mathematics once they got to college (National Science Board, 2010, as cited in Bressoud, 2012). It would seem best to help these students with remediation while in high school, prior to enrolling in college.

It is possible for these students to continue with their AP Calculus studies, supplementing with precalculus review throughout the year. These students, however, would effectively be studying for more than one course, and their performance on calculus materials may suffer due to lack of time studying and lack of understanding due to precalculus shortcomings. Some high schools offer non-AP courses in calculus; these courses are typically taught at a slower pace, which may allow for an easier time reviewing precalculus topics. AP Statistics (or a similar non-AP offering) may be an alternate option instead of AP Calculus for those who want a college-level mathematics course, but McMullin (2012 January 25) noted that AP Statistics "is largely a writing course with very little algebra in it," and that these students will still not be receiving the precalculus remediation that they will need in order to succeed at the college level.

The Squeeze Theorem

The Squeeze Theorem (also known as the Sandwich Theorem or the Pinching Theorem) is not specifically covered in the AP Calculus curriculum, but is sometimes covered in college curricula. It is not typically listed as a topic in textbooks' tables of contents; indeed, out of the eight textbooks examined in this study, only one (Salas et al., 2007) mentioned the Pinching Theorem in its table of contents. There were some course descriptions and final examinations that included this topic, however; it was specifically mentioned by eight of the 50 institutions (16.0%), but only one of the 24 institutions (4.2%) included it on a final examination.

Larson et al. (2006) covered the Squeeze Theorem in their section on methods of evaluating limits analytically. They stated:

> If $h(x) \leq f(x) \leq g(x)$ for all *x* in an open interval containing *c*, except possibly at *c* itself, and if
>
> $$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$
>
> then $\lim_{x \to c} f(x)$ exists and is equal to *L*.

A typical application of the Squeeze Theorem is that it can be used to prove two useful trigonometric limit identities, $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ and $\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$. For those institutions that do not cover l'Hôpital's Rule, these can be very useful facts when solving limit problems that involve trigonometric functions. Indeed, analysis of the 24 institutions with final examinations did show several tests that had limit problems dealing with trigonometric functions; at many of those institutions, however, the curriculum also included l'Hôpital's Rule, which made those identities virtually obsolete.

Those very useful trigonometric limit identities could be stated without proof, so some AP Calculus teachers who wish to introduce them may do so without needing to cover the Squeeze Theorem. Still, there may still be value to covering the Squeeze Theorem, even if assessments in the classroom and the AP Calculus examination do not require it. Going over the Squeeze Theorem with visual representations is typically an easy way for students to comprehend the concept, and going over those limit identities using graphical representations can help to take away the mystery behind those results. Other than that, it may not be necessary to go more in-depth with the concept. Sometimes, students struggle to determine their own bounding functions when trying to apply the Squeeze Theorem to other trigonometric limits; AP Calculus teachers may wish to devote more time to helping students use those two identities to solve those problems, instead. In this way, students have at least been exposed to the Squeeze Theorem, even if they did not deal with more complex problems. Since those types of problems seem to be of little concern to both the AP Calculus examination writers and college final examination writers, it does not appear to be a tremendous loss.

State and/or Prove a Definition or Theorem

Stating and/or proving definitions or theorems is a topic that was not included at all in the AP Calculus curriculum. It was explicitly included in the curricula of six out of 50 institutions (12.0%), and was covered on three out of 24 (12.5%) institutions' final examinations. While not often covered, this is not to say that the details of definitions and theorems are not important. There are times on the AP Calculus examination that students are expected to explain why a certain theorem does or does not apply, and details regarding prerequisite conditions (such as continuity or differentiability) are required. To that extent, AP Calculus teachers may desire to give students older AP Calculus examination problems to try for practice.

Some college institutions, though, do place a greater emphasis on proofs than the AP Calculus program. Some examinations only required students to state a complete definition or theorem, while others required students to write out a complete proof. The subjects of the proofs are dependent on the prerequisite material covered in the calculus course, however. The proofs of the properties of limits, as well as the Squeeze Theorem, require the ε-δ definition of a limit, which is not in the AP Calculus curriculum. Institutions that cover the ε-δ definition of a limit may be more inclined to cover such proofs, and their students would obviously be better prepared for those proofs.

There are other proofs, however, that do not require extra knowledge. In fact, some of these proofs require only basic knowledge or techniques that most AP Calculus students are expected to understand. The proof of the Chain Rule, for example, only requires the definition of the derivative, which is something that AP Calculus students are expected to comprehend prior to discovering the derivative formulas. Proofs of derivative rules like $\frac{d}{dx}\left[\sin(u)\right] = \cos(u)\cdot u'$ only require the definition of the derivative and trigonometric identities. The proof of the statement $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ simply uses the technique of logarithmic differentiation.

There are also proofs of concepts that require other calculus theorems to be used as lemmas. The concept of using concavity to determine whether tangent lines provide underestimates or overestimates is included in the Calculus AB curriculum; this concept can be proven using the Mean Value Theorem. The differentiability and continuity of inverse functions can also be proven, using the definition of a derivative as well as the Intermediate Value

Theorem. The formula for the derivative of an inverse function can easily be derived using the Chain Rule.

As previously stated, Gollub et al. (2002) recommended that AP Calculus teachers look into giving students the opportunities to examine and discover proofs in the AP Calculus classroom. It may not be prudent to examine proofs of every theorem, but teachers may wish to examine different proofs to see which ones might work best in the classroom. For some topics that students struggle with, it may be an opportunity to try to develop better understanding by examining the proofs.

Results and Analysis: Institutions of Note

California Institute of Technology

The California Institute of Technology, often referred to as Caltech, is a private Group I institution (AMS, 2011). It has a selective freshman admissions process, only admitting about 230 new students each fall. One of the admissions requirements for students is that they have taken four years of mathematics courses in high school, with calculus as a stated requirement. Caltech does not grant advanced placement for any AP examinations, and it requires all students to take a placement examination in mathematics prior to enrollment. Based on the results of this placement examination, students may earn advanced standing and credits on a case-by-case basis (California Institute of Technology, 2012). Because Caltech does not accept AP examination scores, and because of its unique course curriculum, it was not included as one of the 50 institutions in this study.

The initial mathematics course for incoming freshmen who do not receive advanced standing is Math 1a, which covers many topics from single-variable calculus. The course description states that the course is not geared toward teaching calculus procedures, since all students should already have taken some sort of calculus course in high school. Instead, the focus of the course is on the "mathematical method," using single-variable calculus as the means to develop mathematical understanding.

Much of the course either covers calculus material at a much deeper depth than the AP Calculus curriculum, or it covers other content as deemed appropriate. The course is heavy in proofs, with the first week of the course spent on basic set theory and different proof methods. The second week of the course is then spent on defining the real number system. The rest of the course covers the topics of single-variable calculus, with most of the course centered on establishing proofs for the different concepts that are covered. Very little focus is spent on calculations, again because students are assumed to have covered those techniques in their high school calculus courses.

Math 1a is a ten-week course, and certain weeks are very noteworthy. Throughout the course, certain topics were covered that are exclusively in the Calculus BC curriculum. These topics included tests for convergence of series, absolute and conditional convergence, and power series (week 3); polar coordinates (week 7); improper integrals and the integral test for series (week 8); approximations, Taylor polynomials, and Taylor series (week 9); and partial fraction decomposition (week 10). The sheer number of topics from the Calculus BC curriculum make it clear that this course may not be appropriate for students who have only taken Calculus AB. Since many of the topics are covered at a more rigorous level, akin to an introductory real analysis course, it would be extremely difficult for students to comprehend these topics without prior exposure and understanding.

The focus of Week 4 was on limits of functions and continuity. Much of the discussion on limits was centered on the ε-δ definition of a limit, which is not an AP Calculus topic, but is sometimes covered in traditional Calculus I courses. Therefore, students who were enrolled in AP Calculus courses that covered the ε-δ definition may have an easier time grasping the nuances of some concepts from this week. Much of the discussion on continuity and its consequences, however, dealt with the topics in terms of set theory and real analysis, much like the Calculus BC topics mentioned above.

The focus of Week 5 was on differential calculus; of all the weeks in Math 1a, this may have been the one that AP Calculus students could have the easiest time grasping and adjusting to. Some of the proofs are based on the definition of the derivative, which is covered in the AP Calculus curriculum. The development of some theorems, such as the Mean Value Theorem, are conducted in a traditional manner like most college calculus textbooks, with Rolle's Theorem established first for use as a lemma to prove the MVT. For AP Calculus instructors who wish to introduce proofs in their course, some of the topics from this week may be appropriate for their high school students.

Because of the selective nature of admissions to the California Institute of Technology, it is highly unlikely that a randomly chosen AP Calculus teacher will have a student that will enroll in Caltech. In the rare scenario where a teacher does get such a student, however, the teacher is advised to consult with the student, to ensure that they understand the different expectations that Caltech has for its students in Math 1a. Those teachers may also want to examine the material found online, to see if they can assist their student with some of the content. At the very least, teacher and student may wish to examine the first two or three weeks of material available online, so that the student has an easier time transitioning to the proof-based style of the course.

Georgia Institute of Technology

The Georgia Institute of Technology, often referred to as Georgia Tech, is a public Group I institution (AMS, 2011). Unlike Caltech, Georgia Tech offers a more traditional 16-week semester course in differential and integral calculus. Two things, however, make Georgia Tech's curriculum unique. The first is its AP Calculus policy—at other schools, students who earn a passing score on the Calculus BC examination typically earn more credits than those who took the Calculus AB examination. At Georgia Tech, however, credit is awarded only for one course, MATH 1501 (Georgia Institute of Technology, 2012). The second thing that makes Georgia Tech's curriculum unique is the sheer number of topics that are covered in that semester. MATH 1501 is somewhat of a mix of the traditional Calculus I and Calculus II courses.

Georgia Tech's MATH 1501 curriculum starts with a review of precalculus topics, but it also lightly touches upon methods of mathematical proofs, including mathematical induction. Based on the course description, it is unclear whether this indicates a course philosophy that is heavy on proofs, but it is possible. It does not appear, however, that such an emphasis would be as heavy as that of Caltech, which was much more like a real analysis course.

Georgia Tech's curriculum does include the ε-δ definition of a limit, which is covered sometime during the first two weeks of the course. Furthermore, Georgia Tech was the only one of the 50 institutions in this study who covered limits of sequences without going over convergence of series; this stand-alone topic is not part of the AP Calculus curriculum, as only limits of functions are considered, and limits of sequences are only needed as a precursor to a discussion on series in Calculus BC. The rest of the coverage of differential calculus seems traditional, with the topics covered similar to those required by the Calculus AB curriculum.

Georgia Tech's coverage of integral calculus is also fairly traditional, although it does cover extra topics at the end when covering advanced techniques and applications. Among the applications that Georgia Tech covers are finding volumes of solids using the shell method, applications of integrals to moments, mass, and center of mass, applications of integrals to work and force, and fluid force (applications of integrals to model physical situations)—none of these topics are covered in the AP Calculus curriculum. Georgia Tech students are also expected to learn advanced antidifferentiation techniques, including antidifferentiation by parts, antidifferentiation using trigonometric substitution, and antidifferentiation by simple partial fractions—all of which are Calculus BC topics.

Because of Georgia Tech's unique policy regarding AP Calculus examination placement, students entering Calculus II may need to review material on their own, should they wish to catch up to their peers who actually took the Calculus I course. For students who took Calculus BC, this is not too difficult—the main shortcoming that would need to be addressed by these students is the coverage of applications of integrals. For students who took Calculus AB, however, they will need to supplement with those techniques of integration that are part of the Calculus BC curriculum. Teachers at high schools with high matriculation to Georgia Tech may wish to supplement their AP Calculus curricula with these topics to help in the transition.

Harvard University

Harvard University is a private Group I institution (AMS, 2011). Math 1a is its Introduction to Calculus course, which is covered in a 15-week semester term. What makes Harvard's calculus course unique is the broad nature of the material covered; while the calculus topics are pretty traditional, the course curriculum includes other topics that are not typically seen in college calculus courses.

With respect to calculus topics, Harvard's curriculum does cover some extra topics. Like many institutions, Harvard covered l'Hôpital's Rule and Newton's Method during its unit on differentiation. It was also one of only three institutions that explicitly stated in its course syllabus that Simpson's Rule would be covered during the course; while this is not an AP Calculus topic, it is often paired in textbooks with the Trapezoidal Rule, which is an AP Calculus topic. As such, some AP Calculus teachers do decide to include Simpson's Rule when covering numerical integration techniques such as rectangular Riemann sums and trapezoidal approximation.

Harvard's curriculum also included additional topics of integration. Advanced antidifferentiation techniques, such as antidifferentiation by parts, antidifferentiation by simple partial fractions, and antidifferentiation using trigonometric substitution were included, all of which are Calculus BC topics. Another Calculus BC topic, improper integrals, was also covered in the Harvard curriculum. Harvard was one of only two institutions to cover applications of integrals to probability, which is not an AP Calculus topic.

Besides the differences in calculus topics, Harvard's curriculum also differed greatly in its focus on various advanced function types that are typically not seen in Calculus I courses, as well as modern mathematical and scientific theories. Harvard's course description mentioned lessons on catastrophe theory, as well as applications of calculus to music and artificial intelligence. Practice final examinations included problems on catastrophe theory and probability, and referenced such functions as the quantum exponential function $\exp_h(x) = (1+h)^{x/h}$ and the sinc function $\operatorname{sinc}(x) = \frac{\sin(x)}{x}$, among others. There were numerous true/false questions, one of which was written as a joke involving the characters Log, Tan, Sin, Cos, and Exp. Matching problems were also commonly used; sometimes, they were used to match graphs to functions, while other times were used to match function names to their characteristics.

Harvard's calculus course somewhat resembles a typical introductory calculus course in some ways, but there are other unique aspects that make it much different, as well. Students who enroll at Harvard should be made aware of this, especially if they are able to skip Math 1a with AP Calculus examination credit. The online explanation of Harvard's AP policy, however, was not entirely clear on how credit is awarded, so it may be possible that students who took AP Calculus courses in high school will still take Math 1a when enrolled at Harvard.

Stanford University

Stanford University is a private Group I institution (AMS, 2011). Stanford is on a quarter system, and students entering with Calculus AB examination scores receive different placements, based on their results. For those who received a score of 4 on the Calculus AB examination, they are allowed to skip Math 41, which is the first of two single-variable calculus courses. For those who received a score of 5 on the Calculus AB examination, however, they are allowed to skip both Math 41 and Math 42 (Stanford University, 2012).

In examining Stanford's Math 41 curriculum, there does not seem to be too many things out of the ordinary. Like other institutions, Stanford adds topics that are not in the Calculus AB curriculum, such as the ε-δ definition of a limit, l'Hôpital's Rule, Newton's Method, and antidifferentiation by parts. Math 42 also does not seem out of the ordinary, as well; it continues with where Math 41 left off, and covers many of the topics that are typically covered in Calculus II courses. Stanford did include some topics that are not in the AP Calculus curriculum at all, such as the various applications of integrals and finding volumes of solids using the shell method.

What is alarming about Stanford's AP Calculus examination policy, however, is that it would allow a student from Calculus AB to pass Math 42, when those students have seen very little of what is covered in the Math 42 course curriculum. Indeed, it would be much more appropriate to have Math 42 waivers granted only for those who passed Calculus BC, as it allows for students who receive a score of 4+ on the Calculus BC examination (Stanford University, 2012). It makes sense that a student who earned a score of 5 on the Calculus AB examination would probably have the knowledge, skills, and capabilities to pass Math 42, but that does not automatically mean that credit should be awarded based solely on the student's potential.

Thus, Calculus AB teachers who have students matriculating to Stanford must be sure to advise their students about their options. While some students might gladly take both credits and never take a mathematics course ever again, those who will be taking courses beyond the single-variable calculus sequence should be strongly recommended to take credit only for Math 41, passing on receiving credit for Math 42 if they are given that option. Otherwise, those students may find themselves too far behind and unable to keep up with the rigors of Math 51, which deals with linear algebra and differentiable multivariable calculus.

University of Chicago

The University of Chicago is a private Group I institution (AMS, 2011). It is on a quarter system, and students entering with a Calculus AB examination score of 5 earn credit for MATH 15100, the school's Calculus I course (University of Chicago, 2012a). The content of the course, as described by the course description, seems very typical for a course in differential calculus. Because it is on a quarter schedule, Chicago does not include any topics from integral calculus in MATH 15100; those topics are reserved for MATH 15200. Chicago does cover the ε-δ definition of a limit in all of its calculus sequences (University of Chicago, 2012b).

While Chicago does have an honors sequence for calculus that is heavy on theory, that is not to say that its typical calculus sequence is lacking in theory. In fact, Chicago's curriculum appeared to have more of a theoretical proof-based focus than most of the institutions in this study. One noteworthy problem from a final examination required students to explain why, for any polynomial of degree *d* with an arbitrary point in the plane, there could be at most *d* tangent lines going through that arbitrary point; since this problem is not scaffolded as to hint to the method of solution, it truly tests the students' ability to problem solve and develop a solution. Another problem asked students to prove that if a function were everywhere differentiable and its first derivative were a polynomial, then the function must be a polynomial; again, the solution is not hinted at. A third question explicitly stated to use proof by induction; while the method was clearly stated, it was still a useful question to determine student understanding of the proof.

Students entering the calculus sequence at the University of Chicago with AP Calculus credit may not necessarily be behind when it comes to exposure to the topics of differential calculus, but they may not be used to the proof-based nature of the course. As such, they may find it difficult to adjust should they jump straight into Calculus II. Since that course starts with the basics of integral calculus, though, this may give those students some time to adjust to the collegiate expectations, as they will initially be covering topics that they had already learned in AP Calculus.

The University of Chicago's program is a typical example of where the AP Calculus program's philosophy and some college mathematics departments' philosophies collide. In the scope of this study, Chicago is but one example, but there are probably other universities out there that teach calculus with slightly more of a proof-based foundation even at the non-honors level. Teachers must be sure to advise their students to check the expectations at their matriculating colleges. If that research is not done, it can lead to a bit of a culture shock and possible struggles in future studies.

University of Pennsylvania

The University of Pennsylvania, often referred to as Penn, is a private Group I institution (AMS, 2011). Penn does not grant advanced placement for the Calculus AB examination; only students who receive a score of 5 on the Calculus BC examination receive credit for its Calculus I course, Math 104 (University of Pennsylvania, 2012a). Because Penn does not accept Calculus AB scores, it was not included as one of the 50 institutions in this study.

While Math 104 is called Calculus I, the content of that course is more typical of a second-semester course. Indeed, Penn also has a course called Introduction to Calculus, Math 103. The topics in this course are more typical of the usual first calculus course, with complete coverage of differential calculus and an introduction to integral calculus. When comparing Penn's Math 103 curriculum to the Calculus AB curriculum, most of the topics align nicely; Penn does, however, include l'Hôpital's Rule (a Calculus BC topic), Newton's Method (not an AP Calculus topic), and hyperbolic functions (not an AP Calculus topic).

Based on a review of the curriculum and recent final examinations, it is puzzling to understand why Penn does not grant Math 103 credit for a Calculus AB score of 4 or 5. The differences in curriculum are few, and would not greatly hinder a student progressing into Math 104. In fact, the final examinations from the last three semesters contain problems that are very much like those on the Calculus AB examination. Altogether, there were 41 problems on three tests—all 38 of the multiple-choice problems could easily have been on the Calculus AB examination, and the three free-response questions certainly were aligned to the Calculus AB curriculum. Furthermore, while the Penn mathematics department does provide opportunities for students to take a department placement examination for almost all of its calculus courses, it specifically prohibits this option for one course—Math 103. The only two options for students to earn Math 103 credit are either to take the course at Penn, or to obtain transfer credit by earning a grade of C+ or better in a calculus course at any other institution (University of Pennsylvania, 2012b).

Thankfully, while Calculus AB students cannot earn any credits for Math 103 to apply toward graduation or major requirements, they are not required to start with Math 103 and suffer through boredom as they go over material that they have already learned in high school. Students who took Calculus AB in high school are allowed to jump straight into Math 104, which covers the rest of integral calculus, as long as they are adequately prepared (University of Pennsylvania, 2012c). This is probably the case for those who earned a 4 or 5 on the Calculus AB examination.

University of Southern California

The University of Southern California, often referred to as USC, is a private Group I institution (AMS, 2011). USC does not grant advanced placement for both the Calculus AB and Calculus BC examinations (University of Southern California, 2012). Because USC does not accept Calculus AB scores, it was not included as one of the 50 institutions in this study.

USC is somewhat unique in that while it does not award course credit for students who took AP Calculus, it does have a special course designed for these students: MATH 127, Enhanced Calculus I, which is offered only in the fall semester. It is intended for students who earned a score of 4 or 5 on the Calculus AB examination, or those who earned a score of 3 or 4 on the Calculus BC examination (University of Southern California, 2011).

The only publicly available information about this course is a short course description in the university's annual catalogue. Based on this description, it seems that this course skips the differential calculus topics that were covered in AP Calculus, focusing primarily on techniques and applications of integration, infinite sequences and series, beginning linear algebra, and ordinary differential equations. Thus, it appears to serve as a bridge to Calculus AB students who still need to learn the materials that were covered by Calculus BC; meanwhile, those Calculus BC students who did not master the material well enough to earn a score of 5 have an opportunity to review the material and strengthen their understanding of it.

USC is not alone in designing courses for students who have already taken AP Calculus in high school. Bressoud (2010d) advocated that this should be a "common practice at every institution with sizeable numbers of students entering with AP credit in calculus." Not many of the 50 institutions surveyed, however, utilized such courses. It is to be seen whether or not more institutions will follow USC's example to create courses like MATH 127.

University of Washington

The University of Washington, often referred to as UW, is a public Group I institution (AMS, 2011). UW is on a quarter system, and students entering with Calculus AB examination scores receive different placements, based on their results. For those who received a score of 3 or 4 on the Calculus AB examination, they are allowed to skip Math 124, which is the first of three single-variable calculus courses. For those who received a score of 5 on the Calculus AB examination, however, they are allowed to skip Math 124 and Math 125 (University of Washington, 2012).

In examining the Math 124 curriculum, it seems to cover the material of a standard differential calculus course. UW does cover logarithmic differentiation as a technique for finding derivatives, and l'Hôpital's Rule is also included in the curriculum. The biggest difference from other institutions is that UW introduces parametric equations throughout the differential calculus course. Analysis of planar curves given in parametric form is a Calculus BC topic, and is typically introduced in the second of a two-semester course. UW is on a quarter system, however, and has rearranged the topics so that all of differential calculus can be covered in the first quarter. A perusal of recent final examinations did show that the questions involving parametric equations were relatively simple, and a student skipping Math 124 at UW with Calculus AB credit could reasonably learn that material and l'Hôpital's Rule on their own.

In examining the Math 125 curriculum, however, there are more gaps in content that require filling. There are several topics from the Calculus BC curriculum that are covered in Math 125, including antidifferentiation by parts, antidifferentiation by simple partial fractions, finding volumes of solids using the shell method, finding the length of a curve (expressed either parametrically or non-parametrically), and improper integrals. There are also several topics that are not included in the AP Calculus curriculum, such as antidifferentiation using trigonometric substitution, applications of integrals to moments, mass, and center of mass, applications of integrals to work and force, and Simpson's Rule.

Like Stanford, it is somewhat alarming that UW would allow a student from Calculus AB to receive credit for Math 125, when those students have not seen much of the content in the Math 125 curriculum. While it would probably be better for those students to take Math 125 at UW, there is also the risk of alienating students, as the course starts off with material that they had already learned in Calculus AB. UW does offer an honors sequence in calculus, but this may not be desirable for those students who only need exposure to calculus for their field of study. Unfortunately, UW does not offer a course for AP Calculus students like that offered by the University of Southern California.

Thus, Calculus AB teachers who have students matriculating to UW must be sure to advise their students about their options. While some students might gladly take both credits and never take a mathematics course ever again, those who will be taking courses beyond Math 124 and Math 125 should be strongly recommended to take credit only for Math 124, passing on receiving credit for Math 125 if they are given that option. Otherwise, those students may find themselves too far behind and unable to keep up with the rigors of Math 126, which deals with sequences, series, and multivariable calculus.

Questions for Future Study

While the results of this study give a glimpse into the differences between the AP Calculus program and some university calculus programs, the study as currently constructed could be expanded to encompass additional institutions across the academic spectrum. With additional time and resources, continued research into Group II, Group III, Group M (highest degree offered of master's in mathematics), and Group B (highest degree offered of bachelor's in mathematics) institutions could lead to the realization of additional trends, as well as possible validation of results from this study. With additional participants, it could also be possible to analyze differences in trends between strata; it is possible that results could vary when data are delineated by factors such as highest degree types, school size, or public/private status.

The study as currently constructed could also be expanded to include an analysis of Calculus BC topics. This study did investigate, to some extent, the value of supplementing the Calculus BC curriculum with topics that are not at all included in the AP Calculus curriculum; this analysis, however, was based solely on the College Board's 2004 data. Examining more recent information from a sample of institutions could help to either validate or discredit the conclusions made in this study. This research could also place a greater focus on those topics that are covered strictly by the Calculus BC curriculum, in order to determine whether or not the level of rigor expected at the university level matches that of the Calculus BC curriculum.

This study could also be expanded to look more deeply into the textbooks that are utilized not only by the different university institutions, but also different by the AP Calculus programs. With additional resources and a survey of textbook usage at the high school and college levels, a collection of complete texts could be analyzed for similarities and differences in content and pedagogy. These differences could have an affect on how each course is taught, including possible topics of focus or neglect in each curriculum.

Ongoing research by Bressoud, Carlson, Pearson, & Rasmussen is focused on identifying, among other things, characteristcs of successful university Calculus I courses (Bressoud, 2009). Early results (Bressoud, 2011a; Bressoud, 2011b) from a stratified sample of 600 institutions provided some demographic information about Calculus I students and instructors. Beginning in the fall of 2012, Bressoud et al. will be conducting case studies of institutions noted to have successful Calculus I programs. Similarly, research could also be conducted to identify the characteristics of successful AP Calculus high school programs. Possible factors (based loosely on factors being examined in Bressoud et al.'s research) could include the educational background and training of the instructor (including participation in week-long summer professional development workshops), use of prior AP Calculus examinations as teaching tools and assessments, the effect of different textbooks, the effect of different graphing calculator models and brands, the effect of technology other than graphing calculators (such as interactive clickers), and the effect of various school schedules (such as length and frequency of class sessions).

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Appendix A: Topic Outline for Calculus BC

The topic outline for Calculus BC includes all Calculus AB topics. Additional topics are found in paragraphs that are marked with a plus sign (+) or an asterisk (*). The additional topics can be taught anywhere in the course that the instructor wishes. Some topics will naturally fit immediately after their Calculus AB counterparts. Other topics may fit best after the completion of the Calculus AB topic outline. (See AP Central for sample syllabi.) Although the exam is based on the topics listed here, teachers may wish to enrich their courses with additional topics.

I. Functions, Graphs, and Limits

Analysis of graphs. With the aid of technology, graphs of functions are often easy to produce. The emphasis is on the interplay between the geometric and analytic information and on the use of calculus both to predict and to explain the observed local and global behavior of a function.

Limits of functions (including one-sided limits)

- An intuitive understanding of the limiting process.
- Calculating limits using algebra.
- Estimating limits from graphs or tables of data.

Asymptotic and unbounded behavior

- Understanding asymptotes in terms of graphical behavior.
- Describing asymptotic behavior in terms of limits involving infinity.
- Comparing relative magnitudes of functions and their rates of change (for example, contrasting exponential growth, polynomial growth, and logarithmic growth).

Continuity as a property of functions

- An intuitive understanding of continuity. (The function values can be made as close as desired by taking sufficiently close values of the domain.)
- Understanding continuity in terms of limits.
- Geometric understanding of graphs of continuous functions (Intermediate Value Theorem and Extreme Value Theorem).

*** Parametric, polar and vector functions.** The analysis of planar curves includes those given in parametric form, polar form and vector form.

II. Derivatives

Concept of the derivative

- Derivative presented graphically, numerically, and analytically.
- Derivative interpreted as an instantaneous rate of change.
- Derivative defined as the limit of the difference quotient.
- Relationship between differentiability and continuity.

Derivative at a point

- Slope of a curve at a point. Examples are emphasized, including points at which there are vertical tangents and points at which there are no tangents.
- Tangent line to a curve at a point and local linear approximation.
- Instantaneous rate of change as the limit of average rate of change.
- Approximate rate of change from graphs and tables of values.

Derivative as a function

- Corresponding characteristics of graphs of f and f'
- Relationship between the increasing and decreasing behavior of f and the sign of f'.
- The Mean Value Theorem and its geometric interpretation.
- Equations involving derivatives. Verbal descriptions are translated into equations involving derivatives and vice versa.

Second derivatives

- Corresponding characteristics of the graphs of f, f', and f''.
- Relationship between the concavity of f and the sign of f''.
- Points of inflection as places where concavity changes.

Applications of derivatives

- Analysis of curves, including the notations of monotonicity and concavity.

\+ Analysis of planar curves in parametric form, polar form and vector form, including velocity and acceleration.

- Optimization, both absolute (global) and relative (local) extrema.
- Modeling rates of change, including related rates problems.
- Use of implicit differentiation to find the derivative of an inverse function.
- Interpretation of the derivative as a rate of change in varied applied contexts, including velocity, speed and acceleration.
- Geometric interpretation of differential equations via slope fields and the relationship between slope fields and solution curves for differential equations.

\+ Numerical solution of differential equations using Euler's method.

\+ L'Hôpital's Rule, including its use in determining limits and convergence of improper integrals and series.

Computation of derivatives

- Knowledge of derivatives of basic functions, including power, exponential, logarithmic, trigonometric and inverse trigonometric functions.
- Derivative rules for sums, products, and quotients of functions.
- Chain rule and implicit differentiation.

\+ Derivatives of parametric, polar and vector functions.

III. Integrals

Interpretations and properties of definite integrals

- Definite integral as a limit of Riemann sums.
- Definite integral of the rate of change of a quantity over an interval interpreted as the change of the quantity over the interval:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

- Basic properties of definite integrals (examples include additivity and linearity).

* **Applications of integrals.** Appropriate integrals are used in a variety of applications to model physical, biological or economic situations. Although only a sampling of applications can be included in any specific course, students should be able to adapt their knowledge and techniques to solve other similar application problems. Whatever applications are chosen, the emphasis is on using the method of setting up an approximating Riemann sum and representing its limit as a definite integral. To provide a common foundation, specific applications should include finding the area of a region (* including a region bounded by polar curves), the volume of a solid with known cross sections, the average value of a function, the distance traveled by a particle along a line, [* the length of a curve (including a curve given in parametric form)], and accumulated change from a rate of change.

Fundamental Theorem of Calculus

- Use of the Fundamental Theorem to evaluate definite integrals.
- Use of the Fundamental Theorem to represent a particular antiderivative, and the analytical and graphical analysis of functions so defined.

Techniques of antidifferentiation

- Antiderivatives following directly from derivatives of basic functions.

\+ Antiderivatives by substitution of variables (including change of limits for definite integrals), [* parts, and simple partial fractions (nonrepeating linear factors only)].

\+ Improper integrals (as limits of definite integrals).

Applications of antidifferentiation

- Finding specific antiderivatives using initial conditions, including applications to motion along a line.
- Solving separable differential equations and using them in modeling (including the study of the equation $y' = ky$ and exponential growth).

\+ Solving logistic differential equations and using them in modeling.

Numerical approximations to definite integrals. Use of Riemann sums (using left, right and midpoint evaluation points) and trapezoidal sums to approximate definite integrals of functions represented algebraically, graphically and by tables of values.

*IV. Polynomial Approximation and Series

*** Concept of series.** A series is defined as a sequence of partial sums, and convergence is defined in terms of the limit of the sequence of partial sums. Technology can be used to explore convergence and divergence.

*** Series of constants**

+ Motivating examples, including decimal expansion.
+ Geometric series with applications.
+ The harmonic series.
+ Alternating series with error bound.
+ Terms of series as areas of rectangles and their relationship to improper integrals, including the integral test and its use in testing the convergence of *p*-series.
+ The ratio test for convergence and divergence.
+ Comparing series to test for convergence or divergence.

*** Taylor series**

+ Taylor polynomial approximation with graphical demonstration of convergence (for example, viewing graphs of various Taylor polynomials of the sine function approximating the sine curve).
+ Maclaurin series and the general Taylor series centered at $x = a$.
+ Maclaurin series for the functions e^x, $\sin x$, $\cos x$ and $\frac{1}{1-x}$.
+ Formal manipulation of Taylor series and shortcuts to computing Taylor series, including substitution, differentiation, antidifferentiation and the formation of new series from known series.
+ Functions defined by power series.
+ Radius and interval of convergence of power series.
+ Lagrange error bound for Taylor polynomials.

Appendix B: AP Calculus Curriculum Research

The table below shows some of the 39 topics listed in the *AP Calculus Curriculum Survey* (College Board, 2004) that are not included in the Calculus AB curriculum, as well as other topics added to the list. (Topics that did not appear on any university curricula were omitted.) Some of the added topics were not originally listed in the survey; other topics are included in the Calculus AB curriculum, but deemphasized in recent examinations.

The "Total" and "Pct" columns indicate the number and percentage of institutions that included that topic on a course description/syllabus, sample final examination, or recent final examination. A grand total of 50 institutions were used in the compilation of this data.

Topic	Total	Pct
L'Hôpital's Rule	39	78.0
Estimation Using Differentials	36	72.0
Precalculus Topics	35	70.0
Newton's Method	20	40.0
Finding Volumes of Solids Using the Shell Method	16	32.0
The ε-δ Definition of a Limit	13	26.0
Logarithmic Differentiation	10	20.0
The Squeeze Theorem	8	16.0
Antidifferentiation by Parts	7	14.0
Applications of Integrals to Work and Force		
Finding the Length of a Curve		
Antidifferentiation by Simple Partial Fractions	6	12.0
Hyperbolic Functions		
State and/or Prove a Definition or Theorem		
Antidifferentiation Using Trigonometric Substitution	5	10.0
Applications of Integrals to Moments, Mass, and Center of Mass	4	8.0
Applications of Integrals in Various Contexts to Model Physical, Biological, or Economic Situations		
Analysis of Planar Curves Given in Parametric Form	3	6.0
Improper Integrals		
Simpson's Rule		
Applications of Integrals to Probability	2	4.0
Finding the Area of a Surface of Revolution		
Numerical Solutions of Differential Equations Using Euler's Method		
Study of Logistic Differential Equations		
Functions Defined by Power Series	1	2.0
Lagrange Error Bound for Taylor Polynomials		
Limits of Sequences		
Maclaurin and Taylor Series		
Radius and Interval of Convergence of Power Series		
Series, Including Geometric, Harmonic, Alternating, and *p*-Series		
Solution Techniques for Non-Separable Differential Equations		
Systems of Differential Equations		
Taylor Polynomial Approximations to Functions		
Tests for Convergence and Divergence		

The table below is similar to the table from the previous page, but only includes institutions that posted a sample final examination or recent final examination. Information gleaned from course descriptions and syllabi were not counted. A grand total of 24 institutions were used in the compilation of this data.

Topic	Total	Pct
Optimization	21	87.5
L'Hôpital's Rule	19	79.2
Modeling Rates of Change	18	75.0
Estimation Using Differentials	12	50.0
Curve Sketching	11	45.8
Logarithmic Differentiation	9	37.5
Newton's Method	8	33.3
Definite Integral as Limit of Riemann Sums	7	29.2
Precalculus Topics	5	20.8
Antidifferentiation by Parts	4	16.7
Applications of Integrals to Work and Force	3	12.5
The ε-δ Definition of a Limit		
Finding the Length of a Curve		
Finding Volumes of Solids Using the Shell Method		
State and/or Prove a Definition or Theorem		
Analysis of Planar Curves Given in Parametric Form	2	8.3
Antidifferentiation by Simple Partial Fractions		
Applications of Integrals to Moments, Mass, and Center of Mass		
Applications of Integrals to Probability		
Comparison Property of the Integral		
Improper Integrals		
Simpson's Rule		
Antidifferentiation Using Trigonometric Substitution	1	4.2
Applications of Integrals in Various Contexts to Model Physical, Biological, or Economic Situations		
Functions Defined by Power Series		
Lagrange Error Bound for Taylor Polynomials		
Limits of Sequences		
Maclaurin and Taylor Series		
Numerical Solutions of Differential Equations Using Euler's Method		
Radius and Interval of Convergence of Power Series		
Series, Including Geometric, Harmonic, Alternating, and *p*-Series		
Solution Techniques for Non-Separable Differential Equations		
The Squeeze Theorem		
Study of Logistic Differential Equations		
Systems of Differential Equations		
Taylor Polynomial Approximations to Functions		
Tests for Convergence and Divergence		

A breakdown by topic (listed in alphabetical order) is provided on the following pages. Each listing also states the source of information (course description or a sample/recent examination).

<u>Analysis of Planar Curves Given in Parametric Form</u>
Texas A&M University (description)
University of Washington (description, examination)
University of Wisconsin-Madison (examination)

<u>Antidifferentiation by Parts</u>
Georgia Institute of Technology (description)
Harvard University (description, examination)
Johns Hopkins University (description)
Stanford University (description, examination)
University of California, Berkeley (examination)
University of Texas at Austin (description)
University of Washington (description, examination)

<u>Antidifferentiation by Simple Partial Fractions</u>
Georgia Institute of Technology (description)
Harvard University (description, examination)
Johns Hopkins University (description)
Stanford University (description)
University of Texas at Austin (description)
University of Washington (description, examination)

<u>Antidifferentiation Using Trigonometric Substitution</u>
Georgia Institute of Technology (description)
Harvard University (description, examination)
Stanford University (description)
University of Texas at Austin (description)
University of Washington (description)

<u>Applications of Integrals in Various Contexts to Model Physical, Biological, or Economic Situations</u>
Bowling Green State University (description)
Georgia Institute of Technology (description)
Southern Illinois University (description)
Stanford University (description, examination)

<u>Applications of Integrals to Moments, Mass, and Center of Mass</u>
Georgia Institute of Technology (description)
University of Houston (description)
University of Washington (description, examination)
University of Wisconsin-Madison (examination)

Applications of Integrals to Probability
Harvard University (description, examination)
Stanford University (description, examination)

Applications of Integrals to Work and Force
Georgia Institute of Technology (description)
Stanford University (examination)
University of Houston (description)
University of Illinois at Urbana-Champaign (description)
University of New Mexico (description)
University of Notre Dame (description, examination)
University of Washington (description, examination)

Comparison Property of the Integral
Stanford University (examination)
University of California, Berkeley (examination)

Curve Sketching
Indiana University Bloomington (examination)
Kansas State University (examination)
Northwestern University (examination)
Stanford University (examination)
Stony Brook University (examination)
University of California, Berkeley (examination)
University of Maryland (examination)
University of Missouri-Kansas City (examination)
University of New Mexico (examination)
University of North Carolina at Charlotte (examination)
University of Washington (examination)

Definite Integral as Limit of Riemann Sums
Stanford University (examination)
Tulane University (examination)
University of California, Berkeley (examination)
University of New Mexico (examination)
University of Notre Dame (examination)
University of Wisconsin-Madison (examination)
Washington University in St. Louis (examination)

The ε-δ Definition of a Limit
Georgia Institute of Technology (description)
Johns Hopkins University (description)
Michigan State University (description)
Rutgers University (description)
Stanford University (examination)
University at Buffalo (description)

The ε-δ Definition of a Limit (continued)
University of Akron (description)
University of California, Berkeley (description, examination)
University of Chicago (description)
University of Hawai‘i at Mānoa (description)
University of Houston (description)
University of Illinois at Chicago (description)
Washington University in St. Louis (examination)

Estimation Using Differentials
Clemson University (description)
Columbia University (description)
Georgia Institute of Technology (description)
Indiana University Bloomington (examination)
Johns Hopkins University (description)
Kansas State University (description)
Michigan State University (description)
New Jersey Institute of Technology (description)
New York University (description)
Northwestern University (examination)
Ohio State University (description)
Pennsylvania State University (description)
Polytechnic Institute of New York University (examination)
Purdue University (description, examination)
Rensselaer Polytechnic Institute (description)
Rutgers University (description)
Southern Illinois University (description)
Stanford University (description, examination)
Texas Tech University (description)
Tulane University (examination)
University at Buffalo (description)
University of Akron (description)
University of California, Berkeley (description)
University of California, San Diego (description)
University of Colorado Denver (description)
University of Hawai‘i at Mānoa (description)
University of Houston (description)
University of Illinois at Chicago (description, examination)
University of Illinois at Urbana-Champaign (description)
University of New Mexico (description, examination)
University of North Carolina at Charlotte (description, examination)
University of Notre Dame (description, examination)
University of South Florida (description)
University of Texas at Austin (description)
University of Washington (description, examination)
Washington University in St. Louis (examination)

Finding the Area of a Surface of Revolution
Bowling Green State University (description)
Johns Hopkins University (description)

Finding the Length of a Curve
Bowling Green State University (description)
Johns Hopkins University (description)
Southern Illinois University (description)
University of Missouri-Kansas City (examination)
University of New Mexico (description)
University of Washington (description, examination)
University of Wisconsin-Madison (examination)

Finding Volumes of Solids Using the Shell Method
Bowling Green State University (description)
Georgia Institute of Technology (description)
Johns Hopkins University (description)
Kansas State University (examination)
Pennsylvania State University (description)
Rensselaer Polytechnic Institute (description)
Southern Illinois University (description)
Stanford University (description, examination)
University of Akron (description)
University of California, Berkeley (description)
University of California, Los Angeles (description)
University of Hawaiʻi at Mānoa (description)
University of Houston (description)
University of Illinois at Urbana-Champaign (description)
University of Notre Dame (description, examination)
University of Washington (description)

Functions Defined by Power Series
Stanford University (description, examination)

Hyperbolic Functions
Purdue University (description)
Southern Illinois University (description)
University of Delaware (description)
University of Illinois at Urbana-Champaign (description)
University of Rhode Island (description)
University of South Florida (description)

Improper Integrals
Harvard University (description)
Stanford University (description, examination)
University of Washington (description, examination)

Lagrange Error Bound for Taylor Polynomials
Stanford University (description, examination)

L'Hôpital's Rule
Clemson University (description)
Columbia University (description)
Duke University (description)
Harvard University (description)
Indiana University Bloomington (examination)
Johns Hopkins University (description)
Kansas State University (examination)
Michigan State University (examination)
New Jersey Institute of Technology (description)
New York University (description)
Northeastern University (description)
Northwestern University (examination)
Ohio State University (description)
Polytechnic Institute of New York University (description, examination)
Purdue University (description, examination)
Rensselaer Polytechnic Institute (description)
Rutgers University (description, examination)
Stanford University (description, examination)
Stony Brook University (description)
Texas A&M University (description)
Texas Tech University (description)
Tulane University (examination)
University at Buffalo (description)
University of California, Berkeley (description, examination)
University of California, San Diego (description, examination)
University of California, Santa Barbara (description)
University of Colorado Denver (description)
University of Illinois at Chicago (description, examination)
University of Illinois at Urbana-Champaign (description)
University of Maryland (examination)
University of Missouri-Kansas City (examination)
University of North Carolina at Charlotte (description, examination)
University of Notre Dame (examination)
University of Rhode Island (description)
University of South Florida (description)
University of Texas at Austin (description)
University of Washington (description, examination)
University of Wisconsin-Madison (examination)
Washington University in St. Louis (examination)

Limits of Sequences
Georgia Institute of Technology (description)

<u>Logarithmic Differentiation</u>
Kansas State University (examination)
Northwestern University (examination)
Purdue University (examination)
Rutgers University (examination)
Southern Illinois University (examination)
Stony Brook University (examination)
Tulane University (examination)
University of Colorado Denver (description)
University of Washington (examination)
University of Wisconsin-Madison (examination)

<u>Maclaurin and Taylor Series</u>
Stanford University (description, examination)

<u>Modeling Rates of Change</u>
Harvard University (examination)
Indiana University Bloomington (examination)
Kansas State University (examination)
Michigan State University (examination)
Northwestern University (examination)
Purdue University (examination)
Rutgers University (examination)
Southern Illinois University (examination)
Stanford University (examination)
Stony Brook University (examination)
Tulane University (examination)
University of California, Berkeley (examination)
University of California, San Diego (examination)
University of Maryland (examination)
University of North Carolina at Charlotte (examination)
University of Notre Dame (examination)
University of Washington (examination)
University of Wisconsin-Madison (examination)

<u>Newton's Method</u>
Bowling Green State University (description)
Columbia University (description)
Harvard University (description)
Michigan State University (description)
New Jersey Institute of Technology (description)
Northwestern University (examination)
Ohio State University (description)
Rutgers University (description)
Stanford University (description, examination)
Stony Brook University (description, examination)

Newton's Method (continued)
Texas A&M University (description)
University of Akron (description)
University of California, Berkeley (examination)
University of Hawaiʻi at Mānoa (description)
University of Houston (description)
University of Illinois at Urbana-Champaign (description)
University of Maryland (description, examination)
University of North Carolina at Charlotte (description, examination)
University of Notre Dame (description, examination)
Washington University in St. Louis (examination)

Numerical Solutions of Differential Equations Using Euler's Method
Duke University (description)
Stanford University (description, examination)

Optimization
Indiana University Bloomington (examination)
Kansas State University (examination)
Michigan State University (examination)
Northwestern University (examination)
Polytechnic Institute of New York University (examination)
Purdue University (examination)
Rutgers University (examination)
Southern Illinois University (examination)
Stanford University (examination)
Stony Brook University (examination)
Tulane University (examination)
University of California, Berkeley (examination)
University of California, San Diego (examination)
University of Illinois at Chicago (examination)
University of Maryland (examination)
University of Missouri-Kansas City (examination)
University of North Carolina at Charlotte (examination)
University of Notre Dame (examination)
University of Washington (examination)
University of Wisconsin-Madison (examination)
Washington University in St. Louis (examination)

Precalculus Topics
Bowling Green State University (description)
Clemson University (description)
Columbia University (description)
Duke University (description)
Georgia Institute of Technology (description)
Harvard University (examination)
Johns Hopkins University (description)
New Jersey Institute of Technology (description)
New York University (description)
Ohio State University (description)
Polytechnic Institute of New York University (description)
Purdue University (description, examination)
Rensselaer Polytechnic Institute (description)
Rutgers University (description)
Southern Illinois University (description)
Stanford University (description)
Stony Brook University (description)
Texas A&M University (description)
Texas Tech University (description)
Tulane University (examination)
University at Buffalo (description)
University of California, Berkeley (examination)
University of California, San Diego (description)
University of California, Santa Barbara (description)
University of Chicago (description, examination)
University of Delaware (description)
University of Hawaiʻi at Mānoa (description)
University of Illinois at Urbana-Champaign (description)
University of Maryland (description)
University of North Carolina at Charlotte (description)
University of Notre Dame (description)
University of Rhode Island (description)
University of South Florida (description)
University of Texas at Austin (description)
University of Washington (description)

Radius and Interval of Convergence of Power Series
Stanford University (description, examination)

Series, Including Geometric, Harmonic, Alternating, and p-Series
Stanford University (description, examination)

<u>Simpson's Rule</u>
Harvard University (examination)
Texas Tech University (description)
University of Washington (examination)

<u>Solution Techniques for Non-Separable Differential Equations</u>
Polytechnic Institute of New York University (examination)

<u>The Squeeze Theorem</u>
Georgia Institute of Technology (description)
Johns Hopkins University (description)
Pennsylvania State University (description)
Stanford University (description)
University of Chicago (description)
University of Colorado Denver (description)
University of Houston (description)
University of North Carolina at Charlotte (examination)

<u>State and/or Prove a Definition or Theorem</u>
Indiana University Bloomington (examination)
Stanford University (examination)
University of California, Los Angeles (description)
University of Chicago (description)
University of Maryland (examination)
University of New Mexico (examination)

<u>Study of Logistic Differential Equations</u>
Duke University (description)
Stanford University (description, examination)

<u>Systems of Differential Equations</u>
Stanford University (description, examination)

<u>Taylor Polynomial Approximations to Functions</u>
Stanford University (description, examination)

<u>Tests for Convergence and Divergence</u>
Stanford University (description, examination)

Appendix C: List of Institution Websites

Below are URLs that list all webpages that were used when gleaning information, listed by institution in alphabetical order.

Bowling Green State University
http://www.bgsu.edu/downloads/cas/file52848.pdf

California Institute of Technology
http://www.math.caltech.edu/~2011-12/1term/ma001a/
http://www.math.caltech.edu/~2011-12/1term/ma001a1/

Clemson University
https://mthsc.clemson.edu/ug/MthSc106/calendar_MWThF_Spring2012.pdf

Columbia University
http://www.math.columbia.edu/department/syllabi/CalcIsyllabus.html

Duke University
http://www.math.duke.edu/first_year/courses/31ls12.pdf

Georgia Institute of Technology
http://www.math.gatech.edu/course/math/1501

Harvard University
http://www.math.harvard.edu/~knill/teaching/math1a_2011/syllabus.html
http://www.math.harvard.edu/~knill/teaching/math1a_2011/final/practice1.pdf
http://www.math.harvard.edu/~knill/teaching/math1a_2011/final/practice2.pdf
http://www.math.harvard.edu/~knill/teaching/math1a_2011/final/final.pdf

Indiana University Bloomington
https://resources.oncourse.iu.edu/access/content/group/08c585e7-240b-4cca-00b5-6c39bc7ca7c7/M211%20Student%20Resources/Departmental%20Final%20Exams%2C%202003-2005/M211-Spring-2008-Final-Exam.pdf
https://resources.oncourse.iu.edu/access/content/group/08c585e7-240b-4cca-00b5-6c39bc7ca7c7/M211%20Student%20Resources/Departmental%20Final%20Exams%2C%202003-2005/M211-2007-Final-Exam.pdf
https://resources.oncourse.iu.edu/access/content/group/08c585e7-240b-4cca-00b5-6c39bc7ca7c7/M211%20Student%20Resources/Departmental%20Final%20Exams%2C%202003-2005/M211FinalExamFall2005.PDF

Johns Hopkins University
http://www.mathematics.jhu.edu/new/undergrad/CourseSyllabi/110.106CalculusISyllabus.pdf
http://www.mathematics.jhu.edu/new/undergrad/CourseSyllabi/110.108CalculusISyllabus.pdf

Kansas State University
http://www.math.ksu.edu/math220/spring-2012/courseinfo.pdf
http://www.math.ksu.edu/courses/exam-archive/math220/220fxs12.pdf
http://www.math.ksu.edu/courses/exam-archive/math220/220fxf11.pdf
http://www.math.ksu.edu/courses/exam-archive/math220/220S11Fin.pdf

Michigan State University
https://www.msu.edu/~krcatov6/us12syllabus132-krcatovich.pdf
http://www.math.msu.edu/CurrentStudents/SampleFinals/mth132.pdf

New Jersey Institute of Technology
http://m.njit.edu/Undergraduate/Course_Syllabi/Summer2012/Math_111-U12.html
http://m.njit.edu/Undergraduate/Course_Syllabi/Spring2012/Math_113-S12.html
http://m.njit.edu/Undergraduate/Course_Syllabi/Summer2012/Math_138-U12.html

New York University
https://sites.google.com/a/nyu.edu/math-ua-121-calculus-i-spring-12/home/calendar

Northeastern University
http://www.math.neu.edu/sites/default/files/syllabi/1341Sp12ZaarourSyll.pdf

Northwestern University
http://www.math.northwestern.edu/courses/exams/214-1_Final-02f.pdf
http://www.math.northwestern.edu/courses/exams/214-1f01.pdf
http://www.math.northwestern.edu/courses/exams/214-1f00.pdf

Ohio State University
http://www.math.osu.edu/files/active/0/151.01.pdf

Pennsylvania State University
http://www.math.psu.edu/files/math140topics.pdf

Polytechnic Institute of New York University
http://www.math.poly.edu/courses/ma1024/MA1024Syllabus.pdf
http://www.math.poly.edu/courses/ma1024/past_exams/MA1024-1324_Final_2003-8-25.pdf
http://www.math.poly.edu/courses/ma1024/past_exams/MA1024-1324_Final_2002-12-19.pdf
http://www.math.poly.edu/courses/ma1024/past_exams/MA1024-1324_Final_2000-12-14.pdf

Purdue University
https://www.math.purdue.edu/academic/files/courses/2011fall/MA16500/165F11assisheetRevised4.pdf
http://www.math.purdue.edu/academic/files/courses/oldexams/165FE-F2010.pdf
http://www.math.purdue.edu/academic/files/courses/oldexams/165FE-F2009.pdf
http://www.math.purdue.edu/academic/files/courses/oldexams/165FE-F2008.pdf

<u>Rensselaer Polytechnic Institute</u>
http://homepages.rpi.edu/~schmid/CALC1/calc1.html

<u>Rutgers University</u>
http://www.math.rutgers.edu/courses/151/151Syllabus/151Spring12syllabus.html
http://www.math.rutgers.edu/courses/151-152/151Review/151reviewfSp12.pdf

<u>Southern Illinois University</u>
http://www.math.siu.edu/syllabii/150-Fall%202007.pdf
http://www.math.siu.edu/Finals/Fall11/150-web.pdf
http://www.math.siu.edu/Finals/112-Spring11/150.pdf
http://www.math.siu.edu/Finals/106-Fall10/150webfnl-106.pdf

<u>Stony Brook University</u>
http://www.math.sunysb.edu/~kirillov/mat131-spr12/index.php?page=syllabus
http://www.math.sunysb.edu/~kirillov/mat131-spr12/practice/final-practice.pdf

<u>Stanford University</u>
http://www.stanford.edu/class/math41/syllabus.html
http://www.stanford.edu/class/math41/oldexams/10final.pdf
http://www.stanford.edu/class/math41/oldexams/09final.pdf
http://www.stanford.edu/class/math41/oldexams/08final.pdf
http://www.stanford.edu/class/math42/schedule.html
http://www.stanford.edu/class/math42/oldexams/w11final.pdf
http://www.stanford.edu/class/math42/oldexams/w10final.pdf
http://www.stanford.edu/class/math42/oldexams/w09final.pdf

<u>Texas A&M University</u>
http://www.math.tamu.edu/courses/math151/currentsched.html

<u>Texas Tech University</u>
http://www.math.ttu.edu/FacultyStaff/Resources/DeptHandbook_current.pdf

<u>Tulane University</u>
http://tulane.edu/sse/math/courses/upload/2008spring_121.pdf
http://tulane.edu/sse/math/courses/upload/2007fall_121.pdf
http://tulane.edu/sse/math/courses/upload/2007spring_121.pdf

<u>University at Buffalo</u>
http://copper.math.buffalo.edu/ugs/syllabi/syllabus_mth141.pdf

<u>University of Akron</u>
http://www.uakron.edu/math/academics/syllabi/221-analytic-geometry-and-calculus-i.dot

University of California, Berkeley
http://math.berkeley.edu/courses/choosing/lowerdivcourses/math1A
http://math.berkeley.edu/sites/default/files/pages/Su09_Final_Exam-A.Adiredja.pdf
http://math.berkeley.edu/sites/default/files/pages/Su09_Final_Exam-B.M.Froehle.pdf
http://math.berkeley.edu/sites/default/files/pages/F08_Final_Exam-I.Agol_.pdf

University of California, Los Angeles
http://www.math.ucla.edu/ugrad/courses/math31ab/31Aoutline.shtml

University of California, San Diego
http://www.math.ucsd.edu/_files/instructor-resources/math_10a_syllabus.pdf
http://www.math.ucsd.edu/~yilou/math10as12/finalsample.pdf
http://www.math.ucsd.edu/_files/instructor-resources/math_20a_syllabus.pdf
http://www.math.ucsd.edu/~mdhyatt/math20a2012sp/sample/finalsample.pdf

University of California, Santa Barbara
http://www.math.ucsb.edu/ugrad/textcompare3A6.htm

University of Chicago
http://math.uchicago.edu/~cskalit/15100_F11/
http://math.uchicago.edu/~cskalit/15100_F11/final.pdf

University of Colorado Denver
http://math.ucdenver.edu/academic/syllabi/topical_syllabi/MATH1401Topics.pdf

University of Delaware
http://www.math.udel.edu/courses/241Syllabus.pdf

University of Hawai‘i at Mānoa
http://math.hawaii.edu/home/system_wide_math/241-251A-2012.pdf

University of Houston
http://www.mathematics.uh.edu/undergraduate/courses/math1431/index.php

University of Illinois at Chicago
http://www.math.uic.edu/coursepages/math180/homework
http://www.math.uic.edu/coursepages/exam_archive/math180/final/m180f11final.pdf
http://www.math.uic.edu/coursepages/exam_archive/math180/final/m180s11final.pdf
http://www.math.uic.edu/coursepages/exam_archive/math180/final/m180f10final.pdf

University of Illinois at Urbana-Champaign
http://www.math.illinois.edu/Bourbaki/Syllabi/syl220.html

University of Maryland, College Park
http://www-math.umd.edu/undergraduate/courses/departmental-course-pages/item/539-math-140-calculus-i.html
http://db.math.umd.edu/testbank/fileget.pl?filenum=3126
http://db.math.umd.edu/testbank/fileget.pl?filenum=3000
http://db.math.umd.edu/testbank/fileget.pl?filenum=2923

University of Missouri-Kansas City
http://cas.umkc.edu/mathematics/MathUGcourses/PreviousExams/M210Sp2008CommonFinal.pdf
http://cas.umkc.edu/Mathematics/MathUGcourses/PreviousExams/M210F2007CommonFinal.pdf

University of New Mexico
http://www.math.unm.edu/~nitsche/courses/162/syll.pdf
http://www.math.unm.edu/~nitsche/courses/162/reviews/reviewfin.pdf

University of North Carolina at Charlotte
http://www.math.uncc.edu/files/curriculum/Math1241.pdf
http://www.math.uncc.edu/files/finals/math1241/spring11.pdf
http://www.math.uncc.edu/files/finals/math1241/Fall10Final.pdf
http://www.math.uncc.edu/files/finals/math1241/CommonFinalExam-MATH1241-Spring2010.pdf

University of Notre Dame
http://www.nd.edu/~apilking/Math10550/Schedule10550F11%20copy.pdf
http://www.nd.edu/~apilking/Math10550/Work/OLd%20Exams/Practice%20Final.pdf
http://www.nd.edu/~apilking/Math10550/Work/OLd%20Exams/FinalF08.pdf
http://www.nd.edu/~apilking/Math10550/Work/OLd%20Exams/finalF07.pdf

University of Pennsylvania
http://www.math.upenn.edu/ugrad/calc/m103/math103syllabus.pdf
http://www.math.upenn.edu/ugrad/calc/m103/exams/103Final-F11-ans.pdf
http://www.math.upenn.edu/ugrad/calc/m103/exams/103finalshrtform11a.pdf
http://www.math.upenn.edu/ugrad/calc/m103/exams/103F10wans.pdf

University of Rhode Island
http://www.math.uri.edu/~merino/summer12/mth141/mth141su12.pdf
http://math.uri.edu/~gfaubert/mth141/summer-1-2012/mth141-1000su2012syl.pdf

University of South Florida
http://math.usf.edu/ug/syllabi/mac2311/

University of Texas at Austin
http://www.ma.utexas.edu/academics/courses/syllabi/M408K.php
http://www.ma.utexas.edu/academics/courses/syllabi/M408C.php

University of Washington
http://www.math.washington.edu/~m124/
http://www.math.washington.edu/~m124/source/quizzes/week10/final_aut11/a11final.pdf
http://www.math.washington.edu/~m124/source/quizzes/week10/final_sp11/sp11final.pdf
http://www.math.washington.edu/~m124/source/quizzes/week10/final_w11/win11final.pdf
http://www.math.washington.edu/~m125/
http://www.math.washington.edu/~m125/Quizzes/week10/125finalA11.pdf
http://www.math.washington.edu/~m125/Quizzes/week10/125finalSp11.pdf
http://www.math.washington.edu/~m125/Quizzes/week10/125finalW11.pdf

University of Wisconsin-Madison
http://math.library.wisc.edu/reserves/221/2011-221-harron-fafin.pdf
http://math.library.wisc.edu/reserves/221/2011-221-passman-fafin.pdf
http://math.library.wisc.edu/reserves/221/2011-221-song-fafin.pdf

Washington University in St. Louis
http://www.math.wustl.edu/~oldexams/EXAMS/math131/m131_FEcF10.pdf
http://www.math.wustl.edu/~oldexams/EXAMS/math131/m131_FEcF09.pdf
http://www.math.wustl.edu/~oldexams/EXAMS/math131/m131_FEcSP09.pdf