Directed Study in Mathematics | Name ______________________

1.1 The Epsilon-Delta Definition of Limit | Date ______________ Pd ______

Required reading:

- Larson 9e: pages 52-53
- Dawkins: Calculus I, section 2-10: The Definition of the Limit
  http://tutorial.math.lamar.edu/Classes/CalcI/DefnOfLimit.aspx
  - Notes: Read from the beginning through the paragraph after Example 2.
    Skip Example 1 and the two paragraphs between Example 1 and Example 2. (Last modified: 05/29/2018)
  - Practice Problems: Review Problems 1-2. (Last modified: 06/04/2018)

Required homework:

- Larson 9e: page 57, problems 43-46
  - Use direct substitution with appropriate notation to find each limit.
- Dawkins: Assignment problems 1-4 (Last modified: 06/04/2018)
  - The value of each limit is already provided, so we do not need to use direct substitution to evaluate.

Additional comments regarding the topic:

To keep things simple in this assignment, we will only deal with finite limits of linear expressions, though Larson and Dawkins both have examples of more advanced problems available for study.

In most epsilon-delta definition of limit problems, the limit value itself is typically easy to calculate. The point here is not *how* to evaluate these limits—we often are able to use direct substitution. In fact, since they are so simple to calculate, these problems sometimes state the value of the limit from the start. Instead, we are focusing on *why* this numerical value is the answer for the limit statement. This epsilon-delta definition gives us a way to mathematically prove this result. It may not seem necessary—after all, these results seem to be basic, like we're trying to prove something as simple as $1+3=4.$ But a rigorous mathematical foundation for the topic of limits actually took mathematicians quite a while to figure out. Just because the result seems very intuitive does not mean the justification is quite so simple.

Many explanations are written in a way that explain the step-by-step *reasoning* behind the proof, but what authors provide is often more than some instructors would expect for a proof. Clean, simple "final" proofs are provided for each example as they are discussed in more detail. For the homework problems, it is perfectly adequate to only provide the "final" proof.

Additional comments regarding the Larson reading:

Larson notes that ε is the lowercase Greek letter *epsilon*, but fails to mention that δ is the lowercase Greek letter *delta*. (You may be familiar with $\Delta,$ which is the *uppercase* Greek letter delta.) That is why we call this "the epsilon-delta definition of limit."

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The statement, "for each $\varepsilon > 0$ there exists a $\delta > 0$ such that" could also be written, "for any positive number ε that we pick, there has to be a related positive number δ such that." Basically, if the limit value is accurately determined, we are saying we can pick any positive value of ε that we want, and there has to be at least one positive value for δ that makes the inequalities true.

That is why, when we read through the explanations of the examples, we typically start with the inequality $|f(x)-L| < \varepsilon,$ substituting in the expression for $f(x)$ and the value for $L.$ Then we perform some algebraic manipulation to this inequality, which yields another inequality that resembles $0 < |x-c| < \delta.$ We see in this process that we start with something involving ε (*for any positive number* ε *that we pick*), and end in a result involving δ (*there has to be a related positive number* δ).

But in performing these algebraic manipulations, we are actually working backwards, in a way. The definition says <u>IF</u> $0 < |x-c| < \delta$ (the statement involving δ) is true, <u>THEN</u> $|f(x)-L| < \varepsilon$ (the statement involving ε) must be true (for the appropriate selection of ε and δ). That last parenthetical is the important part—we are doing all of this algebra in order to reason out *what we should use* for $\delta.$ Once we have done enough work to determine that selection , then we can actually write our proof in the correct manner.

Example 6

This example is not a typical proof problem—it asks us to show that, when we use a specific value $\varepsilon = 0.01$, there has to be a value for δ that we can use to justify the limit result. We end up with $\delta = 0.005$, so we can (if we want to) write a proof that shows if $0 < |x-3| < 0.005$, then $|(2x-5)-1| < 0.01$. This end result hints at the possibility that $\lim_{x\to 3}(2x-5) = 1$, since we have appropriate selections of ε and δ with a true if-then statement. But all of this only works for a *single* value of ε. In order to be a real proof, we need to show such a relationship is true for <u>all</u> positive values of ε. That is why actual proofs start with the abstract statement that $\varepsilon > 0$, without actually giving a number for ε.

That said, this examples still does a good job of showing the algebraic manipulation and the thought process behind an appropriate proof, only with specific numbers for ε and δ in place of abstract variables. We could easily replace 0.01 with ε, and end up with a proper proof, as shown below:

Proof: Given $\varepsilon > 0$, let $\delta = \varepsilon/2$.

$$0 < |x-3| < \delta \Rightarrow 0 < |x-3| < \frac{\varepsilon}{2} \Rightarrow 0 < |2x-6| < \varepsilon \Rightarrow |(2x-5)-1| < \varepsilon$$

Therefore, $\lim_{x\to 3}(2x-5) = 1$.

That said, let us look at the explanation they provide. Note how they start with the expression $|(2x-5)-1|$. This is analogous to the statement $|f(x)-L|$, since we basically have $f(x) = 2x-5$ and $L = 1$. We then want to do some algebraic manipulation—probably something involving factoring—to get an expression that resembles $|x-c|$. That is what we get when we end up with $2|x-3|$, and it nicely agrees with our given $c = 3$.

The rest of the example simply puts that algebraic manipulation together with the inequalities. $|(2x-5)-1| < 0.01$ is analogous to $|f(x)-L| < \varepsilon$, and the resulting statement $|x-3| < 0.005$ is analogous to $0 < |x-c| < \delta$. That is why we *choose* $\delta = 0.005$, as those parts of the expressions match up.

Example 7

Again, we start with the expression $|(3x-2)-4|$, which is analogous to $|f(x)-L|$. We can manipulate this to get $3|x-2|$, which resembles our $|x-c|$ with $c = 2$.

Combining this with the inequality statements, we have $|(3x-2)-4| < \varepsilon$ implies $3|x-2| < \varepsilon$. If we divide both sides of this inequality by 3, we get $|x-2| < \varepsilon/3$. This is analogous to $|x-c| < \delta$, so that is why we *choose* $\delta = \varepsilon/3$. Now that we have determined our value for δ, we can write a proper proof:

Proof: Given $\varepsilon > 0$, let $\delta = \varepsilon/3$.

$$0 < |x-2| < \delta \Rightarrow 0 < |x-2| < \frac{\varepsilon}{3} \Rightarrow 0 < |3x-6| < \varepsilon \Rightarrow |(3x-2)-4| < \varepsilon$$

Therefore, $\lim_{x\to\infty 2}(3x-2) = 4$.

Additional comments regarding the Dawkins reading:

Dawkins uses the variable a instead of c and phrases the last part of the definition in a slightly different way than Larson, but it basically means the same thing: $|f(x)-L|<\varepsilon$ is a true statement *whenever* (i.e., *if*) $0<|x-a|<\delta$ is a true statement.

Dawkins' explanation using the diagram probably makes more sense than Larson's explanation, even though both have similar images. x is sufficiently close to a (i.e., $x\to a$) when x is in the pink region. L is sufficiently close to $f(x)$ (i.e., $f(x)\to L$) when $f(x)$ is in the yellow region.

Example 2

Dawkins writes the definition inequalities on one line, inserting the expression and values in the appropriate places. On a nearby line, he shows the algebraic manipulation that we expect, ending with the statement $|x-2|<\varepsilon/5$ nearly underneath the statement $0<|x-2|<\delta$. This is a nice juxtaposition, showing that the δ and $\varepsilon/5$ are analogous to one another. Hence, we can define $\delta=\varepsilon/5$ and write the proper proof.

Dawkins does a better job than Larson in trying to write a proper proof. It is still a little verbose, but at least it is adequate.

Proof: Given $\varepsilon>0$, let $\delta=\varepsilon/5$.

$$0<|x-2|<\delta \Rightarrow 0<|x-2|<\frac{\varepsilon}{5} \Rightarrow 0<|5x-10|<\varepsilon \Rightarrow |(5x-4)-6|<\varepsilon$$

Therefore, $\lim_{x\to 2}(5x-4)=6$.