§9.8 Velocity and Acceleration

Velocity and Acceleration

Displacement and Total Distance

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Learning Goals: Students will be able to…

- Describe the velocity and acceleration associated with a vector-valued function.
- Describe the horizontal and vertical movement of a particle with position defined by a vector-valued function.

Velocity and Acceleration

We are now ready to combine our study of parametric equations, curves, vectors, and vector-valued functions to form a model for motion along a curve. We will begin by looking at the motion of an object in the plane.

As an object moves along a curve in the plane, the coordinates x and y of its position are each functions of time t. Rather than using the letters f and g to represent these two functions, it is convenient to write $x = x(t)$ and $y = y(t)$. So the position vector $\mathbf{r}(t)$ takes the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$.

The beauty of this vector model for representing motion is that we can use the first and second derivatives of the vector-valued function $\mathbf{r}$ to find the object's **velocity** and **acceleration**, both of which are vector quantities having magnitude and direction. Moreover, the magnitude of the velocity vector gives the **speed** of the object at time t.

Velocity and Acceleration

DEFINITIONS OF VELOCITY AND ACCELERATION

If x and y are twice-differentiable functions of t, and $\mathbf{r}$ is a position vector given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, then the velocity vector, acceleration vector, and speed at time t are as follows.

$$\textbf{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t)\rangle$$

$$\textbf{Acceleration} = \mathbf{a}(t) = \mathbf{r}''(t) = \langle x''(t), y''(t)\rangle$$

$$\textbf{Speed} = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

An object with position vector $\mathbf{r}$ is at rest when both components of velocity—that is, both $x'(t)$ and $y'(t)$—are equal to zero.

Example: Velocity and Acceleration

The position vector $\mathbf{r}(t) = \langle 2\cos(t),\, 2\sin(t)\rangle$ describes the path of an object moving in the plane. Find the velocity vector, speed, and acceleration vector of the object.

Example: Velocity and Acceleration

A particle is moving along the curve given by $\mathbf{r}(t) = \langle t^3 - 6t^2,\, -t^2 + 8t\rangle$ for $t \geq 0$.

(a) Is the particle moving to the left or to the right at time $t = 2$? Justify your answer.

(b) At what value(s) of t is the particle at rest? Justify your answer.

Displacement and Total Distance

Recall from section 4.4c that we used integration to find the displacement of a particle and the total distance traveled. When the motion of the particle is described using vector-valued functions, use the following formulas.

DISTANCE AND TOTAL DISPLACEMENT

Let $\mathbf{v}(t) = \langle x'(t), y'(t) \rangle$ be the velocity of a particle, where $x'(t)$ and $y'(t)$ are continuous on $[a, b]$. The **displacement** of the particle on $[a, b]$ is

$$\int_a^b \mathbf{v}(t)\, dt = \left\langle \int_a^b x'(t)\, dt, \int_a^b y'(t)\, dt \right\rangle$$

and the **total distance traveled** by the particle on $[a, b]$ is

$$\int_a^b \|\mathbf{v}(t)\|\, dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

Example: Displacement and Total Distance

A particle moves with velocity $\mathbf{v}(t) = \langle 2t+1, 5 \rangle$. If the particle is at the point (1, 2) at time $t = 0$, find its position when $t = 3$. Then find the total distance traveled on the interval $0 \le t \le 3$.

Example: Displacement and Total Distance

Given $\mathbf{v}(t) = \langle 4t, 3t^2 \rangle$ and $\mathbf{r}(2) = \langle 1, 5 \rangle$, find the position vector.

Example: Displacement and Total Distance

An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where $\frac{dx}{dt} = \tan(e^{-t})$ and $\frac{dy}{dt} = \sec(e^{-t})$ for $t \ge 0$. At time $t = 1$, the object is at position $(2, -3)$.

(a) Find the acceleration vector and the speed of the object at time $t = 1$.

(b) Find the total distance traveled by the object over the time interval $1 \le t \le 2$.