§9.7 Vector-Valued Functions

Plane Curves and Vector-Valued Functions

Differentiation of Vector-Valued Functions

Integration of Vector-Valued Functions

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
Calculus, AP Edition, 9th ed. by Larson & Edwards. © 2010 Brooks/Cole, Cengage Learning.

Learning Goals: Students will be able to…

- Differentiate a vector-valued function.
- Integrate a vector-valued function.

Learning Objectives: Students will be able to…

2.1C Calculate derivatives.

2.1D Determine higher order derivatives.

3.1A Recognize antiderivatives of basic functions.

3.3B Calculate antiderivatives, and evaluate definite integrals.

3.5A Analyze differential equations to obtain general and specific solutions.

Plane Curves and Vector-Valued Functions

In section 9.2, a plane curve was defined as the set of ordered pairs $(f(t), g(t))$ together with their defining parametric equations $x = f(t)$ and $y = g(t)$, where f and g are continuous functions of t on an interval I.

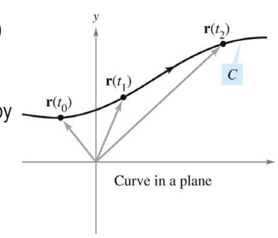
Another way to represent a plane curve is with a **vector-valued function**. This type of function maps real numbers to vectors and is of the form $\mathbf{r}(t) = \langle f(t), g(t) \rangle$ where the **component functions** f and g are real-valued functions of the parameter t.

In typeset material, vectors are usually denoted by lowercase, boldface letters such as **r**. When written by hand, however, vectors are often denoted by letters with arrows above them, such as $\vec{r}$.

Plane Curves and Vector-Valued Functions

Vector-valued functions serve dual roles in the representation of curves. By letting the parameter t represent time, we can use a vector-valued function to represent *motion* along a curve. Or, in the more general case, we can use a vector-valued function to *trace the graph* of a curve.

In either case, the terminal point of the position vector $\mathbf{r}(t)$ coincides with the point (x, y) on the curve given by the parametric equations, as shown in the figure. The arrowhead on the curve indicates the curve's orientation by pointing in the direction of increasing values of t.



Curve in a plane

Unless otherwise stated, the **domain** of a vector-valued function **r** is considered to be the intersection of the domains of the component functions f and g.

Differentiation of Vector-Valued Functions

The definition of the derivative of a vector-valued function parallels the definition for real-valued functions.

DEFINITION OF THE DERIVATIVE OF A VECTOR-VALUED FUNCTION

The **derivative of a vector-valued function r** is defined by

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

for all t for which the limit exists. If $\mathbf{r}'(t)$ exists, then **r** is **differentiable at *t*.** If $\mathbf{r}'(t)$ exists for all t in an open interval I, then **r** is **differentiable on the interval *I*.** Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

Differentiation of Vector-Valued Functions

Differentiation can be done on a *component-by-component basis*. This important result is listed in the next theorem. Note that the derivative of the vector-valued function **r** is itself a vector-valued function.

THEOREM DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS

If $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, where f and g are differentiable functions of t, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t) \rangle.$$

Higher-order derivatives of vector-valued functions are obtained by successive differentiation of each component function.

Example: Differentiation of Vector-Valued Functions

Find the second derivative of the vector-valued function $\mathbf{r}(t) = \left\langle t^2 + t,\ t^3 - 3t^2 \right\rangle$.

Integration of Vector-Valued Functions

The next definition is a consequence of the definition of the derivative of a vector-valued function.

DEFINITION OF INTEGRATION OF VECTOR-VALUED FUNCTIONS

If $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, where f and g are continuous on $[a, b]$, then the **indefinite integral (antiderivative)** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\, dt = \left\langle \int f(t)\, dt,\ \int g(t)\, dt \right\rangle$$

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\, dt = \left\langle \int_a^b f(t)\, dt,\ \int_a^b g(t)\, dt \right\rangle.$$

Integration of Vector-Valued Functions

The antiderivative of a vector-valued function is a family of vector-valued functions all differing by a constant vector $\mathbf{C}$.

For instance, if $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, then for the indefinite integral $\int \mathbf{r}(t)\, dt$, we obtain two constants of integration $\int f(t)\, dt = F(t) + C_1$ and $\int g(t)\, dt = G(t) + C_2$, where $F'(t) = f(t)$ and $G'(t) = g(t)$.

These two *scalar* constants produce one *vector* constant of integration.

$$\int \mathbf{r}(t)\, dt = \langle F(t) + C_1,\ G(t) + C_2 \rangle = \langle F(t), G(t) \rangle + \langle C_1, C_2 \rangle = \mathbf{R}(t) + \mathbf{C}$$

where $\mathbf{R}'(t) = \mathbf{r}(t)$.

Example: Integration of Vector-Valued Functions

Find the indefinite integral: $\int \left\langle \cos(t),\ t^3 - 6t \right\rangle dt$

Example: Integration of Vector-Valued Functions

Evaluate the definite integral: $\int_{-1}^{1} \left\langle t^2,\ \sqrt[5]{t} \right\rangle dt$

Example: Integration of Vector-Valued Functions

Given $\mathbf{r}'(t) = \left\langle \dfrac{1}{1+t^2},\ \dfrac{1}{t} \right\rangle$, find $\mathbf{r}(t)$ that satisfies the initial condition $\mathbf{r}(1) = \langle 2, 0 \rangle$.