

§9.5 Area in Polar Coordinates

Area of a Polar Region

Points of Intersection of Polar Graphs

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
Calculus, AP Edition, 9th ed. by Larson & Edwards. © 2010 Brooks/Cole, Cengage Learning.

Learning Goals: Students will be able to...

- Find the area of a region bounded by a polar graph.
- Find the points of intersection of two polar graphs.

Learning Objectives: Students will be able to...

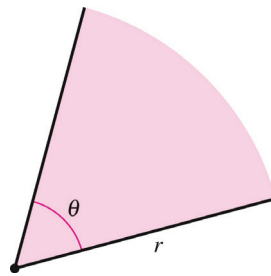
- 2.1C Calculate derivatives.
- 3.1A Recognize antiderivatives of basic functions.
- 3.3B Calculate antiderivatives, and evaluate definite integrals.
- 3.4D Apply definite integrals to problems involving area, volume, (BC: and length of a curve).

Area of a Polar Region

The development of a formula for the area of a polar region parallels that for the area of a region on the rectangular coordinate system, but uses sectors of a circle instead of rectangles as the basic elements of area.

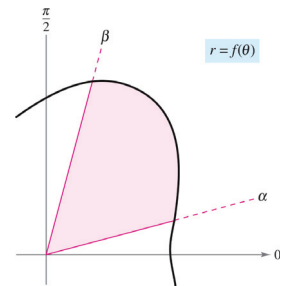
In the figure at the right, note that the area of a circular sector of radius r is $\frac{1}{2}\theta r^2$, provided θ is in radians.

This is because the area of the entire circle would be $A = \pi r^2$; taking a fraction of this circle would yield a fraction of the area: $\frac{\theta}{2\pi} A = \frac{\theta}{2\pi} \pi r^2 = \frac{1}{2}\theta r^2$.



Area of a Polar Region

Consider the function $r = f(\theta)$, where f is continuous and nonnegative on the interval $\alpha \leq \theta \leq \beta$. The region bounded by the graph of f and the radial lines $\theta = \alpha$ and $\theta = \beta$ is shown in the figure.



Area of a Polar Region

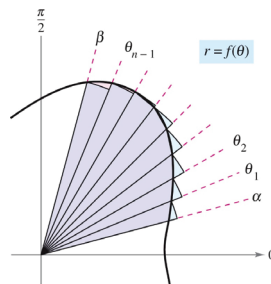
To find the area of this region, partition the interval $[\alpha, \beta]$ into n equal subintervals.

Then approximate the area of the region by the sum of the n sectors, as shown in the figure.

radius of i th sector = $f(\theta_i)$

central angle of i th sector = $\frac{\beta - \alpha}{n} = \Delta\theta$

$$A \approx \sum_{i=1}^n \frac{1}{2} \Delta\theta [f(\theta_i)]^2$$

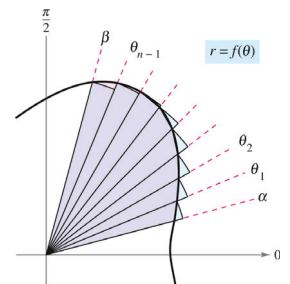


Area of a Polar Region

Taking the limit as $n \rightarrow \infty$ produces

$$A = \lim_{n \rightarrow \infty} \frac{1}{2} \sum_{i=1}^n [f(\theta_i)]^2 \Delta\theta$$

$$= \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$$



Area of a Polar Region

THEOREM AREA IN POLAR COORDINATES

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

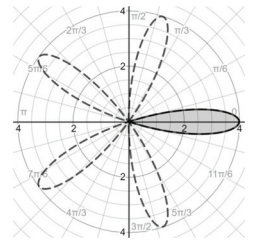
$$A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$$

$$= \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta. \quad 0 < \beta - \alpha \leq 2\pi$$

Useful formulas: $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ $\cos^2(x) = \frac{1 + \cos(2x)}{2}$

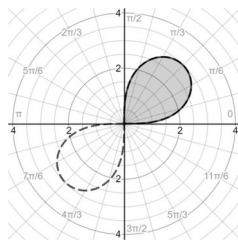
Example: Area of a Polar Region

Find the area of one petal of $r = 4\cos(5\theta)$.



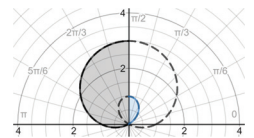
Example: Area of a Polar Region

Find the area of the interior of $r^2 = 9\sin(2\theta)$.



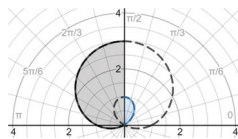
Example: Area of a Polar Region

Find the area of the region between the loops of $r = 1 + 2\sin(\theta)$.



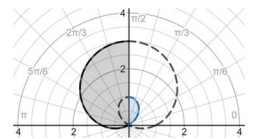
Example: Area of a Polar Region

Find the area of the region between the loops of $r = 1 + 2\sin(\theta)$.



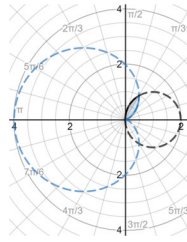
Example: Area of a Polar Region

Find the area of the region between the loops of $r = 1 + 2\sin(\theta)$.



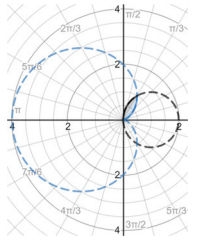
Example: Points of Intersection of Polar Graphs

Find the area of the region common to the interiors of $r = 2\cos(\theta)$ and $r = 2 - 2\cos(\theta)$.



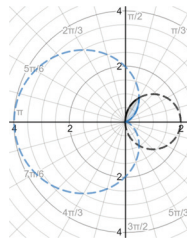
Example: Points of Intersection of Polar Graphs

Find the area of the region common to the interiors of $r = 2\cos(\theta)$ and $r = 2 - 2\cos(\theta)$.



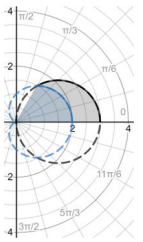
Example: Points of Intersection of Polar Graphs

Find the area of the region common to the interiors of $r = 2\cos(\theta)$ and $r = 2 - 2\cos(\theta)$.



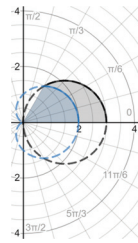
Example: Points of Intersection of Polar Graphs

Find the area of the region inside $r = 3\cos(\theta)$ and outside $r = 1 + \cos(\theta)$.



Example: Points of Intersection of Polar Graphs

Find the area of the region inside $r = 3\cos(\theta)$ and outside $r = 1 + \cos(\theta)$.



Example: Points of Intersection of Polar Graphs

Find the area of the region inside $r = 3\cos(\theta)$ and outside $r = 1 + \cos(\theta)$.

