§9.5 Area in Polar Coordinates

Area of a Polar Region

Points of Intersection of Polar Graphs

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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Learning Goals: Students will be able to…

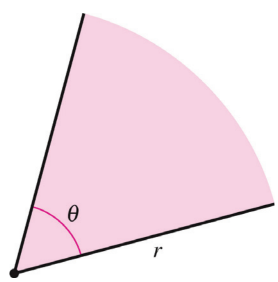
- Find the area of a region bounded by a polar graph.
- Find the points of intersection of two polar graphs.

Area of a Polar Region

The development of a formula for the area of a polar region parallels that for the area of a region on the rectangular coordinate system, but uses sectors of a circle instead of rectangles as the basic elements of area.

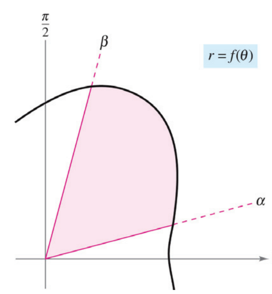
In the figure at the right, note that the area of a circular sector of radius r is $\frac{1}{2}\theta r^2$, provided θ is in radians.

This is because the area of the entire circle would be $A = \pi r^2$; taking a fraction of this circle would yield a fraction of the area: $\frac{\theta}{2\pi}A = \frac{\theta}{2\pi}\pi r^2 = \frac{1}{2}\theta r^2$.



Area of a Polar Region

Consider the function $r = f(\theta)$, where f is continuous and nonnegative on the interval $\alpha \le \theta \le \beta$. The region bounded by the graph of f and the radial lines $\theta = \alpha$ and $\theta = \beta$ is shown in the figure.



Area of a Polar Region

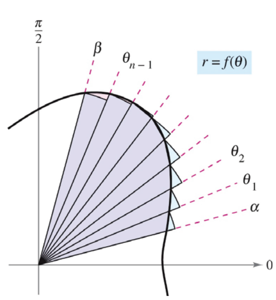
To find the area of this region, partition the interval $[\alpha, \beta]$ into n equal subintervals.

Then approximate the area of the region by the sum of the n sectors, as shown in the figure.

$$\text{radius of } i\text{th sector} = f(\theta_i)$$

$$\text{central angle of } i\text{th sector} = \frac{\beta - \alpha}{n} = \Delta\theta$$

$$A \approx \sum_{i=1}^{n} \frac{1}{2}\Delta\theta\left[f(\theta_i)\right]^2$$

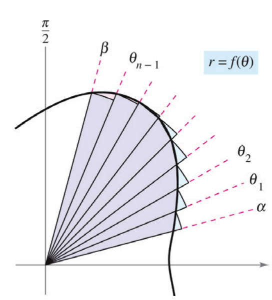


Area of a Polar Region

Taking the limit as $n \to \infty$ produces

$$A = \lim_{n\to\infty} \frac{1}{2}\sum_{i=1}^{n}\left[f(\theta_i)\right]^2 \Delta\theta$$

$$= \frac{1}{2}\int_{\alpha}^{\beta}\left[f(\theta)\right]^2 d\theta$$



Area of a Polar Region

THEOREM **AREA IN POLAR COORDINATES**

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

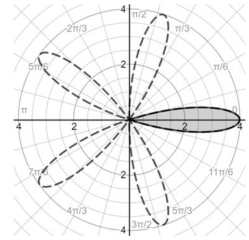
$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

$$= \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta. \qquad 0 < \beta - \alpha \leq 2\pi$$

Useful formulas: $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$ $\qquad \cos^2(x) = \dfrac{1+\cos(2x)}{2}$

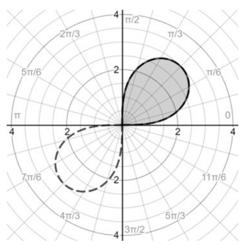
Example: Area of a Polar Region

Find the area of one petal of $r = 4\cos(5\theta)$.



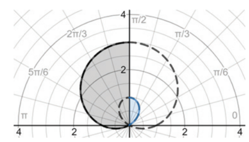
Example: Area of a Polar Region

Find the area of the interior of $r^2 = 9\sin(2\theta)$.



Example: Area of a Polar Region

Find the area of the region between the loops of $r = 1+2\sin(\theta)$.



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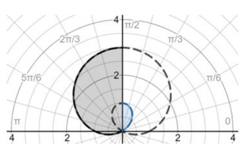

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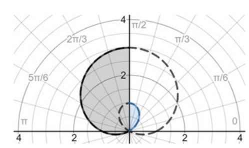
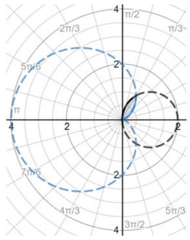


Example: Points of Intersection of Polar Graphs

Find the area of the region common to the interiors of $r = 2\cos(\theta)$ and $r = 2 - 2\cos(\theta)$.



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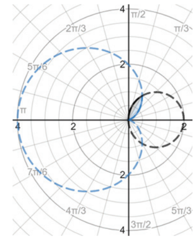

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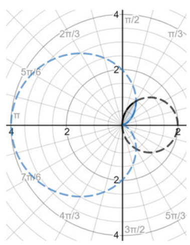
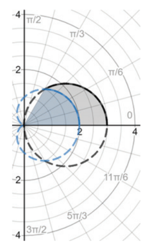


Example: Points of Intersection of Polar Graphs

Find the area of the region inside $r = 3\cos(\theta)$ and outside $r = 1 + \cos(\theta)$.



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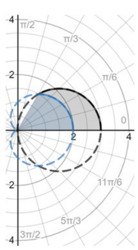

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