§9.4a Polar Coordinates and Polar Graphs

Polar Coordinates

Coordinate Conversion

Special Polar Graphs

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Learning Goals: Students will be able to…

- Understand the polar coordinate system.
- Rewrite rectangular coordinates in polar form and vice versa.
- Sketch the graph of an equation given in polar form.
- Identify several types of special polar graphs.

Polar Coordinates

So far, we have been representing graphs as collections of points (x, y) on the rectangular coordinate system. The corresponding equations for these graphs have been in rectangular form.

In this chapter, we will study a coordinate system called the **polar coordinate system**.

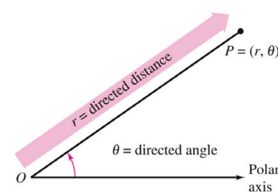
Polar Coordinates

To form the polar coordinate system in the plane, fix a point O, called the **pole** (or **origin**), and construct from O an initial ray called the **polar axis**, as shown in the figure.

Then each point P in the plane can be assigned **polar coordinates** (r, θ) as follows:

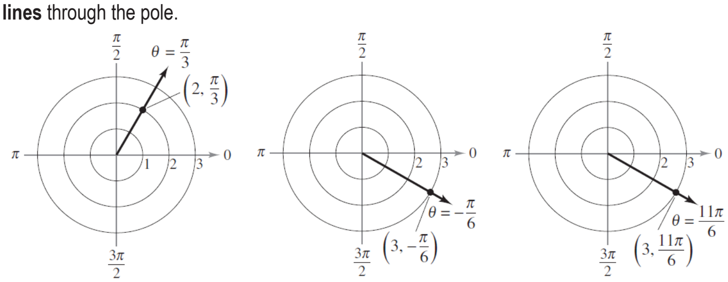
$r =$ directed distance from O to P

$\theta =$ directed angle, counterclockwise from polar axis to segment $\overline{OP}$



Polar Coordinates

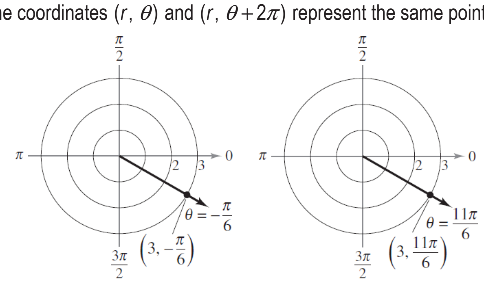
The figure shows three points on the polar coordinate system. Notice that in this system, it is convenient to locate points with respect to a grid of concentric circles intersected by **radial lines** through the pole.



Polar Coordinates

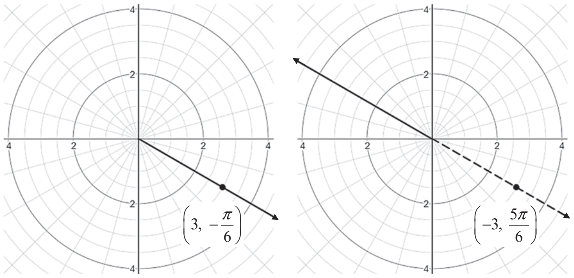
With rectangular coordinates, each point (x, y) has a unique representation. This is not true with polar coordinates.

For instance, the coordinates (r, θ) and $(r, \theta + 2\pi)$ represent the same point.



Polar Coordinates

Also, because r is a *directed distance*, the coordinates $(r,\ \theta)$ and $(-r,\ \theta+\pi)$ represent the same point.



Polar Coordinates

In general, the point $(r,\ \theta)$ can be written as

$$(r,\ \theta)=(r,\ \theta+2n\pi)$$

or

$$(r,\ \theta)=(-r,\ \theta+(2n+1)\pi)$$

where n is any integer.

Moreover, the pole is represented by $(0,\ \theta)$, where θ is any angle.

Coordinate Conversion

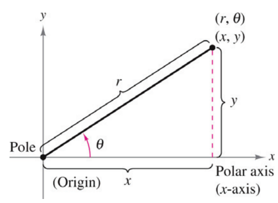
To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive x-axis and the pole with the origin, as shown in the figure.

Because $(x,\ y)$ lies on a circle of radius r, it follows that $r^2=x^2+y^2$.

Moreover, for $r>0$, the definitions of the trigonometric functions imply that

$$\tan(\theta)=\frac{y}{x},\qquad \cos(\theta)=\frac{x}{r},\qquad \sin(\theta)=\frac{y}{r}.$$

The same relationships also hold for $r<0$.



Coordinate Conversion

THEOREM COORDINATE CONVERSION

The polar coordinates $(r,\ \theta)$ of a point are related to the rectangular coordinates $(x,\ y)$ of the point as follows.

1. $x = r\cos\theta$ **2.** $\tan\theta = \frac{y}{x}$

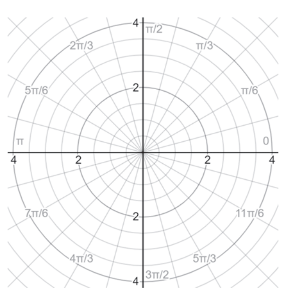
$y = r\sin\theta$ $r^2 = x^2 + y^2$

The first set of equations is used to convert from polar coordinates to rectangular coordinates.

The second set of equations is used to convert from rectangular coordinates to polar coordinates.

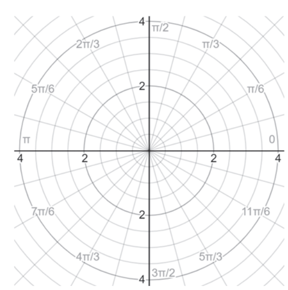
Example: Coordinate Conversion

Plot the point $(-2,\ 5\pi/3)$ in polar coordinates and find the corresponding rectangular coordinates for the point.



Example: Coordinate Conversion

Given the point $(\sqrt{3},\ -1)$, plot the point and find *two* sets of polar coordinates for the point for $0\le\theta<2\pi$.



Special Polar Graphs

Several important types of graphs have equations that are simpler in polar form than in rectangular form.

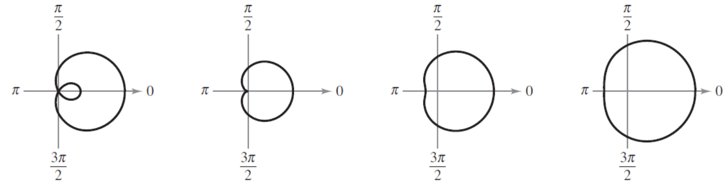
For example, the polar equation of a circle having a radius of a and centered at the origin is simply $r = a$.

Several other types of graphs that have simpler equations in polar form are shown as follows:

Special Polar Graphs

Limaçons: $r = a \pm b\cos(\theta)$ or $r = a \pm b\sin(\theta)$, where $a > 0,\ b > 0$

$0 \le \theta \le 2\pi$ traces out the entire curve



$\frac{a}{b} < 1$ Limaçon with inner loop

$\frac{a}{b} = 1$ Cardioid (heart-shaped)

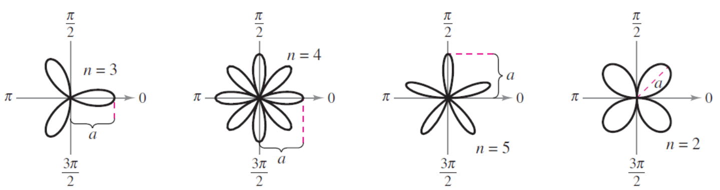
$1 < \frac{a}{b} < 2$ Dimpled limaçon

$\frac{a}{b} \ge 2$ Convex limaçon

Special Polar Graphs

Rose Curves: n petals when n is odd, $2n$ petals when n is even ($n \ge 2$)

$0 \le n\theta \le \pi$ (sine) or $-\pi/2 \le n\theta \le \pi/2$ (cosine) traces out one petal



$r = a\cos n\theta$ Rose curve

$r = a\cos n\theta$ Rose curve

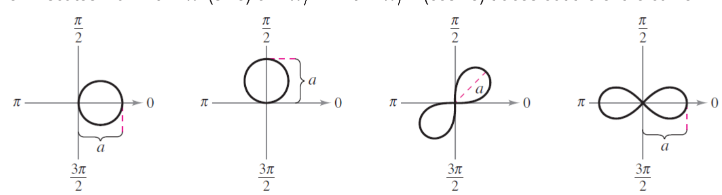
$r = a\sin n\theta$ Rose curve

$r = a\sin n\theta$ Rose curve

Special Polar Graphs

Circles: $0 \le \theta \le \pi$ traces out the entire curve

Lemniscates: $0 \le 2\theta \le \pi$ (sine) or $-\pi/2 \le 2\theta \le \pi/2$ (cosine) traces out the entire curve



$r = a\cos\theta$ Circle

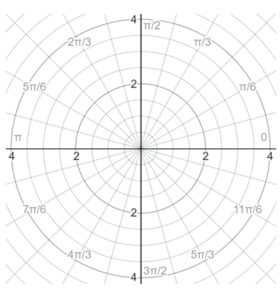
$r = a\sin\theta$ Circle

$r^2 = a^2\sin 2\theta$ Lemniscate

$r^2 = a^2\cos 2\theta$ Lemniscate

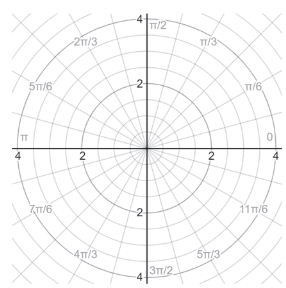
Example: Special Polar Graphs

Sketch a graph of the polar equation $r = 1 + 2\sin(\theta)$.



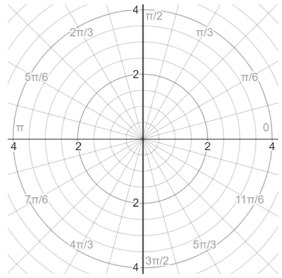
Example: Special Polar Graphs

Sketch one petal of the graph of the polar equation $r = 4\sin(3\theta)$.



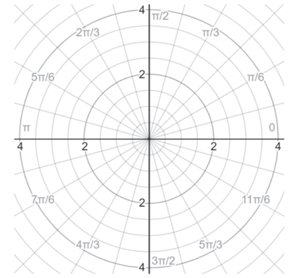
Example: Special Polar Graphs

Sketch one petal of the graph of the polar equation $r = 3\cos(2\theta)$.



Example: Special Polar Graphs

Sketch a graph of the polar equation $r = 3\cos(\theta)$.



Example: Special Polar Graphs

Sketch a graph of the polar equation $r^2 = 16\sin(2\theta)$.