§9.3 Parametric Equations and Calculus

The First and Second Derivatives

Net Change Theorem

Arc Length

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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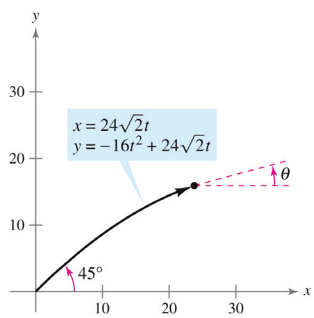
Learning Goals: Students will be able to…

- Find the slope of a tangent line to a curve given by a set of parametric equations.
- Determine intervals on which a curve given by a set of parametric equations is concave upward or downward.
- Utilize the Net Change Theorem to determine the position of a particle described with parametric equations.
- Find the arc length of a curve given by a set of parametric equations.

The First and Second Derivatives

Now that we can represent a graph in the plane by a set of parametric equations, it is natural to ask how to use calculus to study plane curves.

Consider the projectile represented by the parametric equations $x = 24\sqrt{2}\,t$ and $y = -16t^2 + 24\sqrt{2}\,t$. From the discussion in section 9.2, we know that these equations enable us to locate the position of the projectile at a given time. We also know from the figure that the object is initially projected at an angle of 45°, or a slope of $m = \tan(45°) = 1$.



But how can we find the slope at some other time t? The next theorem answers this question by giving a formula for the slope of the tangent line as a function of t.

The First and Second Derivatives

THEOREM PARAMETRIC FORM OF THE DERIVATIVE

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{dx}{dt} \neq 0.$$

dx/dt and dy/dt can also be written as $x'(t)$ and $y'(t)$, respectively.

But note that dy/dx is $y'(x)$.

The First and Second Derivatives

Because dy/dx is a function of t, we can use this theorem repeatedly to find *higher-order* derivatives.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt} \qquad \text{Second derivative}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{\frac{d}{dt}\left[\frac{d^2y}{dx^2}\right]}{dx/dt}. \qquad \text{Third derivative}$$

Example: The First and Second Derivatives

Given $x = t^2 + 5t + 4,\ y = 4t$, find dy/dx and d^2y/dx^2.

Then find the slope and concavity (if possible) at $t = 0$.

Example: The First and Second Derivatives

Given $x = t^4 + 2,\ y = t^3 + t$, find an equation of the tangent line at the point $(3,\ -2)$.

Example: The First and Second Derivatives

Given $x = t^2 - 4,\ y = t^3 - 3t,\ t \ge 0$, find all points (if any) of horizontal and vertical tangency to the curve.

Example: The First and Second Derivatives

Given $x = t^2,\ y = \ln(t)$, determine the open t-intervals on which the curve is concave upward or concave downward.

Net Change Theorem

THEOREM THE NET CHANGE THEOREM

The definite integral of the rate of change of a quantity $F'(x)$ gives the total change, or **net change,** in that quantity on the interval $[a, b]$.

$$\int_a^b F'(x)\,dx = F(b) - F(a) \qquad \text{Net change of } F$$

We learned the Net Change Theorem in section 4.4c.

Net Change Theorem

We can rearrange this equation to get:

$$F(b) = F(a) + \int_a^b F'(x)\,dx$$

This rearranged equation shows that, if we have a function $F'(x)$ and a value $F(a)$, we can obtain $F(b)$. This is especially useful in calculator-active problems when $F'(x)$ is a function that is not easily antidifferentiated.

Similar formulas for motion problems include:

$$s(b) = s(a) + \int_a^b s'(t)\,dt \qquad v(b) = v(a) + \int_a^b v'(t)\,dt$$

where $s'(t)$ is the velocity function and $v'(t)$ is the acceleration function.

Example: Net Change Theorem

An object moving along a curve in the xy-plane is at position $(x(t),\ y(t))$ at time t with $\dfrac{dx}{dt} = \arctan\left(\dfrac{t}{1+t}\right)$ and $\dfrac{dy}{dt} = \ln(t^2 + 1)$ for $t \ge 0$. At time $t = 0$, the object is at position $(-3,\ -4)$. Find $x(4)$.

Arc Length

We have seen how parametric equations can be used to describe the path of a particle moving in the plane. We will now develop a formula for determining the *distance* traveled by the particle along its path.

Arc Length

Recall from section 6.4 that the formula for the arc length of a curve C given by $y = h(x)$ over the interval $[x_0, x_1]$ is $s = \int_{x_0}^{x_1} \sqrt{1 + [h'(x)]^2}\, dx = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.

If C is represented by the parametric equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$, and if $dx/dt = f'(t) > 0$, then:

$$s = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, dx = \int_a^b \sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}}\, \frac{dx}{dt}\, dt$$

$$= \int_a^b \frac{\sqrt{(dx/dt)^2 + (dy/dt)^2}}{dx/dt}\, \frac{dx}{dt}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Arc Length

THEOREM **ARC LENGTH IN PARAMETRIC FORM**

If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \le t \le b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

When applying the arc length formula to a curve, be sure that the curve is traced out only once in the interval of integration.

Example: Arc Length

Find the arc length of the curve $x = 6t^2$, $y = 2t^3$ on the interval $1 \le t \le 4$.