§9.2 Plane Curves and Parametric Equations

Plane Curves and Parametric Equations

Eliminating the Parameter

Finding Parametric Equations

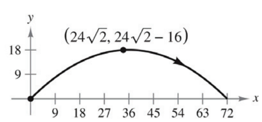
Learning Goals: Students will be able to…

- Sketch the graph of a curve given by a set of parametric equations.
- Eliminate the parameter in a set of parametric equations.
- Find a set of parametric equations to represent a curve.

Plane Curves and Parametric Equations

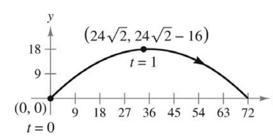
Until now, we have been representing a graph by a single equation involving two variables (either x and y, or r and θ). In this chapter, we will study situations in which three variables are used to represent a curve in the plane.

Consider the path followed by an object that is propelled into the air at an angle of 45°. For an initial velocity of 48 feet per second, the object travels the parabolic path given by $y = -\frac{x^2}{72} + x$, as shown in the figure.



Plane Curves and Parametric Equations

This equation, however, does not tell the whole story. Although it does tell us *where* the object was at a given point (x, y), it does not tell us *when* the object was at a given point (x, y).



To determine this time, we can introduce a third variable t, called a **parameter**. By writing both x and y as functions of t, we obtain the **parametric equations** $x = 24\sqrt{2}\,t$ and $y = -16t^2 + 24\sqrt{2}\,t$.

From this set of equations, we can determine that at time $t = 0$, the object is at the point (0, 0). Similarly, at time $t = 1$, the object is at the point $(24\sqrt{2}, 24\sqrt{2} - 16)$, and so on.

For this particular motion problem, x and y are continuous functions of t, and the resulting path is called a **plane curve**.

Plane Curves and Parametric Equations

DEFINITION OF A PLANE CURVE

If f and g are continuous functions of t on an interval I, then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are called **parametric equations** and t is called the **parameter.** The set of points (x, y) obtained as t varies over the interval I is called the **graph** of the parametric equations. Taken together, the parametric equations and the graph are called a **plane curve,** denoted by C.

When sketching a curve represented by a set of parametric equations, we can plot points in the xy-plane. Each set of coordinates (x, y) is determined from a value chosen for the parameter t. By plotting the resulting points in order of increasing values of t, the curve is traced out in a specific direction. This is called the **orientation** of the curve.

Example: Plane Curves and Parametric Equations

Sketch the curve represented by the parametric equations $x = 1 + \frac{1}{t}$, $y = t - 1$ (indicate the orientation of the curve).

Example: Plane Curves and Parametric Equations

Sketch the curve represented by the parametric equations $x = 8\cos(\theta),\ y = 8\sin(\theta)$ (indicate the orientation of the curve).

Eliminating the Parameter

Finding a rectangular equation that represents the graph of a set of parametric equations is called **eliminating the parameter**. For instance, we can eliminate the parameter from a set of parametric equations as follows.

Parametric equations	⇨	Solve for t in one equation.	⇨	Substitute into second equation.	⇨	Rectangular equation
$x = t^2 - 4$ $y = t/2$		$t = 2y$		$x = (2y)^2 - 4$		$x = 4y^2 - 4$

The range of x and y implied by the parametric equations may be altered by the change to rectangular form. In such instances, the domain of the rectangular equation must be adjusted so that its graph matches the graph of the parametric equations.

Example: Eliminating the Parameter

Write the rectangular equation corresponding to the parametric equations $x = 1 + \frac{1}{t},\ y = t - 1$ by eliminating the parameter.

Example: Eliminating the Parameter

Write the rectangular equation corresponding to the parametric equations $x = 8\cos(\theta),\ y = 8\sin(\theta)$ by eliminating the parameter.

Finding Parametric Equations

The previous examples illustrate techniques for sketching the graph represented by a set of parametric equations. We will now investigate the reverse problem. How can we determine a set of parametric equations for a given graph or a given physical description? Note that such a representation is not unique.

Example: Finding Parametric Equations

Find two different sets of parametric equations for the rectangular equation $y = \frac{4}{x-1}$.

Example: Finding Parametric Equations

Find a set of parametric equations for the rectangular equation $y = 2x - 5$ that satisfies the condition $t = 0$ at the point (3, 1).