§8.8b Varying-Sign Series

Absolute and Conditional Convergence

Strategies for Testing Series

Interval of Convergence and Endpoint Convergence

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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Taylor Polynomials and Infinite Series by B. Goldstein. © 2015 Benjamin Goldstein.

Learning Goals: Students will be able to…

- Classify a convergent series as absolutely or conditionally convergent.
- Review the tests for convergence and divergence of an infinite series.
- Find the radius and interval of convergence of a power series.
- Determine the endpoint convergence of a power series.

Learning Objectives: Students will be able to…

1.1A Express limits symbolically using correct notation, and interpret limits expressed symbolically.

1.1C Determine limits of functions.

4.1A *(BC: Determine whether a series converges or diverges.)*

4.2C *(BC: Determine the radius and interval of convergence of a power series.)*

Absolute and Conditional Convergence

Occasionally, a series may have both positive and negative terms and not be an alternating series.

For instance, the series $\sum_{n=1}^{\infty}\frac{\sin(n)}{n^2}=\frac{\sin(1)}{1}+\frac{\sin(2)}{4}+\frac{\sin(3)}{9}+\cdots$ has both positive and negative terms, yet it is not an alternating series.

Absolute and Conditional Convergence

One way to obtain some information about the convergence of this series is to investigate the convergence of the series $\sum_{n=1}^{\infty}\left|\frac{\sin(n)}{n^2}\right|$.

For all $n\geq 1$, $0<\left|\frac{\sin(n)}{n^2}\right|<\frac{1}{n^2}$.

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series with $p=2>1$.

Therefore, $\sum_{n=1}^{\infty}\left|\frac{\sin(n)}{n^2}\right|$ converges by the Direct Comparison Test.

The next theorem tells us that the original series also converges.

Absolute and Conditional Convergence

THEOREM ABSOLUTE CONVERGENCE

If the series $\Sigma|a_n|$ converges, then the series $\Sigma\, a_n$ also converges.

The converse of this theorem is not true: Just because $\sum a_n$ converges, that does not necessarily guarantee that $\sum|a_n|$ also converges.

For instance, the **alternating harmonic series** $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ converges by the Alternating Series Test. Yet the harmonic series $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right|=\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges. This type of convergence is called **conditional**.

Absolute and Conditional Convergence

DEFINITIONS OF ABSOLUTE AND CONDITIONAL CONVERGENCE

1. $\Sigma\, a_n$ is **absolutely convergent** if $\Sigma\,|a_n|$ converges.
2. $\Sigma\, a_n$ is **conditionally convergent** if $\Sigma\, a_n$ converges but $\Sigma\,|a_n|$ diverges.

THEOREM ABSOLUTE CONVERGENCE

If the series $\Sigma|a_n|$ converges, then the series $\Sigma\, a_n$ also converges.

Example: Absolute and Conditional Convergence

Determine whether the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+1}$ converges absolutely or conditionally, or diverges.

Example: Absolute and Conditional Convergence

Determine whether the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+1}$ converges absolutely or conditionally, or diverges.

Example: Absolute and Conditional Convergence

Determine whether the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$ converges absolutely or conditionally, or diverges.

Strategies for Testing Series

We have now studied several tests for determining the convergence or divergence of an infinite series. Skill in choosing and applying the various tests will come only with practice. Below is a set of guidelines for choosing an appropriate test.

1. Does the nth term approach 0? If not, the series diverges. (nth-Term Test)
2. Is the series one of the special types—geometric, p-series, or alternating?
3. Can the Integral Test or Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?
   (Direct Comparison Test, Limit Comparison Test)

Interval of Convergence and Endpoint Convergence

THEOREM **CONVERGENCE OF A POWER SERIES**

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$.
3. The series converges absolutely for all x.

The number R is the **radius of convergence** of the power series. If the series converges only at c, the radius of convergence is $R = 0$, and if the series converges for all x, the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

Interval of Convergence and Endpoint Convergence

The Ratio Test was used to determine the radius of convergence. The same result can be used to determine an interval for which the series converges.

But this is not necessarily the *interval of convergence*. To determine this, we must also test for convergence at each endpoint.

Example: Interval of Convergence and Endpoint Convergence

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(3x)^n}{4^n + 1}$.

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Example: Interval of Convergence and Endpoint Convergence

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n}$.

Example: Interval of Convergence and Endpoint Convergence

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n}$.