§8.7b Positive-Term Series

Direct Comparison Test

Limit Comparison Test

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
Calculus, AP Edition, 9th ed. by Larson & Edwards. © 2010 Brooks/Cole, Cengage Learning.
Taylor Polynomials and Infinite Series by B. Goldstein. © 2015 Benjamin Goldstein.

Learning Goals: Students will be able to…

- Use the Direct Comparison Test to determine whether a series converges or diverges.
- Use the Limit Comparison Test to determine whether a series converges or diverges.

Learning Objectives: Students will be able to…

1.1A Express limits symbolically using correct notation, and interpret limits expressed symbolically.

1.1C Determine limits of functions.

4.1A *(BC: Determine whether a series converges or diverges.)*

Direct Comparison Test

For the convergence tests developed so far, the terms of the series have to be fairly simple and the series must have special characteristics in order for the convergence tests to be applied.

A slight deviation from these special characteristics can make a test nonapplicable. For example, in the pairs listed below, the second series cannot be tested by the same convergence test as the first series, even though it is similar to the first.

- $\sum_{n=0}^{\infty}\frac{1}{2^n}$ is geometric, but $\sum_{n=0}^{\infty}\frac{n}{2^n}$ is not.
- $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a *p*-series, but $\sum_{n=1}^{\infty}\frac{1}{n^3+1}$ is not.
- $a_n=\frac{n}{(n^2+3)^2}$ is easily integrable, but $b_n=\frac{n!}{(n^2+3)^2}$ is not.

Direct Comparison Test

THEOREM DIRECT COMPARISON TEST

Let $0 < a_n \le b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

This test allows us to *compare* a series having complicated terms with a simpler series whose convergence or divergence is known.

Direct Comparison Test

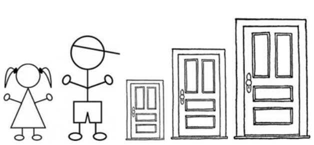
Both parts of the Direct Comparison Test require that $0 < a_n \le b_n$. An informal way of visualizing the Direct Comparison Test is shown in the diagram below.

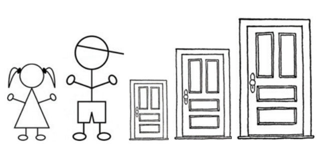
Let the "smaller" series a_n be represented by the shorter person Anna.

Let the "larger" series b_n be represented by the taller person Brian.

"Convergence" means the person can fit through the door.

"Divergence" means the person cannot fit through the door.



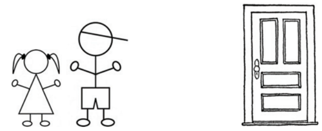


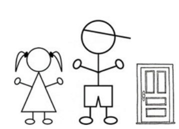
Direct Comparison Test

Left: If the taller Brian can fit through the door, then the shorter Anna also can fit.

Right: If the shorter Anna cannot fit through the door, then the taller Brian also cannot fit.

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.





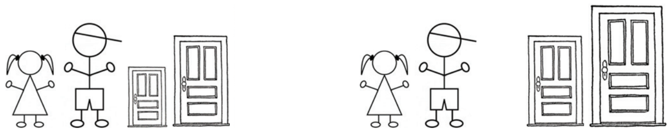
Direct Comparison Test

Left: If the taller Brian cannot fit through the door, we cannot determine whether the shorter Anna can fit through the door.

Right: If the shorter Anna can fit through the door, we cannot determine whether the taller Brian can fit through the door.

If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ may converge or diverge.

If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ may converge or diverge.



Example: Direct Comparison Test

Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{5^n+1}{2^n}$.

Example: Direct Comparison Test

Determine the convergence or divergence of the series $\sum_{n=6}^{\infty} \frac{n}{2^n+1}$.

Limit Comparison Test

Sometimes a series closely resembles a p-series or a geometric series, yet we cannot establish the term-by-term comparison necessary to apply the Direct Comparison Test.

Under these circumstances, we may be able to apply a second comparison test, called the **Limit Comparison Test**.

THEOREM **LIMIT COMPARISON TEST**

Suppose that $a_n > 0$, $b_n > 0$, and

$$\lim_{n \to \infty} \left(\frac{a_n}{b_n} \right) = L$$

where L is *finite and positive*. Then the two series $\Sigma\, a_n$ and $\Sigma\, b_n$ either both converge or both diverge.

Example: Limit Comparison Test

Determine the convergence or divergence of the series $\sum_{n=2}^{\infty} \frac{n^2-1}{n^3}$.

Example: Limit Comparison Test

Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{5^n-2}$.