§8.7a The Integral Test and *p*-Series

The Integral Test

p-Series and Harmonic Series

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
Calculus, AP Edition, 9th ed. by Larson & Edwards. © 2010 Brooks/Cole, Cengage Learning.
Taylor Polynomials and Infinite Series by B. Goldstein. © 2015 Benjamin Goldstein.

Learning Goals: Students will be able to…

- Use the Integral Test to determine whether an infinite series converges or diverges.
- Use properties of p-series and harmonic series.

Learning Objectives: Students will be able to…

1.1A Express limits symbolically using correct notation, and interpret limits expressed symbolically.
1.1C Determine limits of functions.
2.1C Calculate derivatives.
3.1A Recognize antiderivatives of basic functions.
3.2D *(BC: Evaluate an improper integral or show that an improper integral diverges.)*
3.3B Calculate antiderivatives, and evaluate definite integrals.
4.1A *(BC: Determine whether a series converges or diverges.)*

The Integral Test

THEOREM THE INTEGRAL TEST

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

The convergence or divergence of $\sum a_n$ is not affected by deleting the first N terms.

Similarly, when the conditions for the Integral Test are satisfied for all $x \geq N > 1$, we can simply use the integral $\int_N^{\infty} f(x)\,dx$ to test for convergence or divergence.

Example: The Integral Test

Determine the convergence or divergence of the series $\sum_{n=2}^{\infty} \frac{2n}{n^2+1}$.

Example: The Integral Test

Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

p-Series and Harmonic Series

A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ is a ***p*-series**, where p is a positive constant.

For $p = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is the **harmonic series**.

Do not confuse p-series with geometric series. Note the differences with these two examples:

p-series: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

Geometric series: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$

p-Series and Harmonic Series

THEOREM **CONVERGENCE OF *p*-SERIES**

The *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

1. converges if $p > 1$, and
2. diverges if $0 < p \leq 1$.

The harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, is a divergent *p*-series with $p = 1 \leq 1$.

Example: *p*-Series and Harmonic Series

Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{\sqrt[4]{n}}{n}$ and $\sum_{n=1}^{\infty} n^{-4/3}$.