

§8.4 Lagrange Remainder

Remainder of a Taylor Polynomial
Lagrange Error Bound

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
Calculus, AP Edition, 9th ed. by Larson & Edwards. © 2010 Brooks/Cole, Cengage Learning.
Taylor Polynomials and Infinite Series by B. Goldstein. © 2015 Benjamin Goldstein.

Learning Goals: Students will be able to...

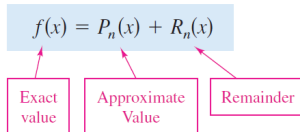
- Determine the Lagrange error bound of a Taylor polynomial.
- Use the Lagrange error bound of a Taylor polynomial.

Learning Objectives: Students will be able to...

- 2.1C Calculate derivatives.
2.1D Determine higher order derivatives.
4.1B (BC: Determine or estimate the sum of a series.)
4.2A (BC: Construct and use Taylor polynomials.)

Remainder of a Taylor Polynomial

An approximation technique is of little value without some idea of its accuracy. To measure the accuracy of approximating a function value $f(x)$ by the Taylor polynomial $P_n(x)$, we can use the concept of a **remainder** $R_n(x)$, defined as follows:



So $R_n(x) = f(x) - P_n(x)$. The absolute value of $R_n(x)$ is called the **error** associated with the approximation.

$$\text{error} = |R_n(x)| = |f(x) - P_n(x)|$$

Remainder of a Taylor Polynomial

The next theorem gives a general procedure for estimating the remainder associated with a Taylor polynomial. This important theorem is called **Taylor's Theorem**, and the remainder given in the theorem is called the **Lagrange form of the remainder**.

Remainder of a Taylor Polynomial

THEOREM TAYLOR'S THEOREM

If a function f is differentiable through order $n + 1$ in an interval I containing c , then, for each x in I , there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}.$$

Lagrange Error Bound

One useful consequence of Taylor's Theorem is called the **Lagrange error bound**:

If $\max|f^{(n+1)}(z)|$ is the maximum value of $f^{(n+1)}(z)$ between x and c , then

$$|R_n(x)| \leq \frac{\max|f^{(n+1)}(z)|}{(n+1)!} |x - c|^{n+1}.$$

When applying Taylor's Theorem, we should not expect to be able to find the exact value of z . (If we could do this, an approximation would not be necessary.)

Rather, we are trying to find bounds for $f^{(n+1)}(z)$ from which we are able to tell how large the remainder $R_n(x)$ is.

Example: Lagrange Error Bound

Use a third-degree Maclaurin polynomial for $f(x) = \sin(x)$ to approximate $\sin(0.5)$. Estimate the error of this approximation.

Example: Lagrange Error Bound

Use a second-degree Maclaurin polynomial for $f(x) = e^x$ to approximate e . Estimate the error of this approximation, and give bounds for the value of e .

Example: Lagrange Error Bound

For a particular function, it is known that $f(2) = 5$, $f'(2) = 8$, and $|f''(x)| \leq 4$ for all x -values in the domain of f . Use a first-degree Taylor polynomial centered at $x = 2$ to approximate $f(2.5)$. Estimate the error of this approximation, and give bounds for the value of $f(2.5)$.

Example: Lagrange Error Bound

Determine the degree of the Maclaurin polynomial required for the error in the approximation of the function $f(x) = e^x$ at $x = 1/2$ to be less than 0.001.

Example: Lagrange Error Bound

Determine the values of x for which the function $f(x) = \cos(x)$ can be replaced by its fourth-degree Maclaurin polynomial if the error cannot exceed 0.001.