§8.4 Lagrange Remainder

Remainder of a Taylor Polynomial

Lagrange Error Bound

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
Calculus, AP Edition, 9th ed. by Larson & Edwards. © 2010 Brooks/Cole, Cengage Learning.
Taylor Polynomials and Infinite Series by B. Goldstein. © 2015 Benjamin Goldstein.

Learning Goals: Students will be able to…

- Determine the Lagrange error bound of a Taylor polynomial.
- Use the Lagrange error bound of a Taylor polynomial.

Learning Objectives: Students will be able to…

2.1C Calculate derivatives.

2.1D Determine higher order derivatives.

4.1B *(BC: Determine or estimate the sum of a series.)*

4.2A *(BC: Construct and use Taylor polynomials.)*

Remainder of a Taylor Polynomial

An approximation technique is of little value without some idea of its accuracy. To measure the accuracy of approximating a function value $f(x)$ by the Taylor polynomial $P_n(x)$, we can use the concept of a **remainder** $R_n(x)$, defined as follows:

$$f(x) = P_n(x) + R_n(x)$$

Exact value — Approximate Value — Remainder

So $R_n(x) = f(x) - P_n(x)$. The absolute value of $R_n(x)$ is called the **error** associated with the approximation.

$$\text{error} = |R_n(x)| = |f(x) - P_n(x)|$$

Remainder of a Taylor Polynomial

The next theorem gives a general procedure for estimating the remainder associated with a Taylor polynomial. This important theorem is called **Taylor's Theorem**, and the remainder given in the theorem is called the **Lagrange form of the remainder**.

Remainder of a Taylor Polynomial

THEOREM TAYLOR'S THEOREM

If a function f is differentiable through order $n+1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

Lagrange Error Bound

One useful consequence of Taylor's Theorem is called the **Lagrange error bound**:

If $\max\left|f^{(n+1)}(z)\right|$ is the maximum value of $f^{(n+1)}(z)$ between x and c, then

$$|R_n(x)| \le \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}|x-c|^{n+1}.$$

When applying Taylor's Theorem, we should not expect to be able to find the exact value of z. (If we could do this, an approximation would not be necessary.)

Rather, we are trying to find bounds for $f^{(n+1)}(z)$ from which we are able to tell how large the remainder $R_n(x)$ is.

Example: Lagrange Error Bound

Use a third-degree Maclaurin polynomial for $f(x) = \sin(x)$ to approximate $\sin(0.5)$.
Estimate the error of this approximation.

Example: Lagrange Error Bound

Use a second-degree Maclaurin polynomial for $f(x) = e^x$ to approximate e.
Estimate the error of this approximation, and give bounds for the value of e.

Example: Lagrange Error Bound

For a particular function, it is known that $f(2) = 5$, $f'(2) = 8$, and $|f''(x)| \le 4$ for all x-values in the domain of f. Use a first-degree Taylor polynomial centered at $x = 2$ to approximate $f(2.5)$.
Estimate the error of this approximation, and give bounds for the value of $f(2.5)$.