§8.3 A Systematic Approach to Taylor Polynomials

Taylor and Maclaurin Polynomials

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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Taylor Polynomials and Infinite Series by B. Goldstein. © 2015 Benjamin Goldstein.

Learning Goals: Students will be able to…

- Find polynomial approximations of elementary functions and compare them with the elementary functions.
- Find Taylor and Maclaurin polynomial approximations of elementary functions.

Taylor and Maclaurin Polynomials

The polynomial approximation of $f(x)=e^x$ in section 8.2 is expanded about $c=0$. For expansions about an arbitrary value of c, it is convenient to write the polynomial in the form $P_n(x)=a_0+a_1(x-c)+a_2(x-c)^2+a_3(x-c)^3+a_4(x-c)^4+\cdots+a_n(x-c)^n$.

In this form, repeated differentiation produces

$$P_n'(x)=a_1+2a_2(x-c)+3a_3(x-c)^2+4a_4(x-c)^3+\cdots+na_n(x-c)^{n-1}$$
$$P_n''(x)=2a_2+2(3a_3)(x-c)+3(4a_4)(x-c)^2+\cdots+n(n-1)a_n(x-c)^{n-2}$$
$$P_n'''(x)=2(3a_3)+2(3)(4a_4)(x-c)+\cdots+n(n-1)(n-2)a_n(x-c)^{n-3}$$
$$P_n^{(4)}=2(3)(4a_4)+\cdots+n(n-1)(n-2)(n-3)a_n(x-c)^{n-4}$$
$$\vdots$$
$$P_n^{(n)}(x)=n(n-1)(n-2)\cdots(2)(1)a_n$$

Taylor and Maclaurin Polynomials

Letting $x=c$, we then obtain

$$P_n(c)=a_0,\quad P_n'(c)=a_1,\quad P_n''(c)=2a_2,$$
$$P_n'''(c)=2(3a_3),\quad P_n^{(4)}(c)=2(3)(4a_4),\quad \ldots,\quad P_n^{(n)}(c)=n!a_n.$$

Because the values of f and its first n derivatives must agree with the values of P_n and its first n derivatives at $x=c$, it follows that

$$f(c)=a_0,\ f'(c)=a_1,\ f''(c)=2a_2,\ f'''(c)=2(3a_3),\ f^{(4)}(c)=2(3)(4a_4),\ \ldots,\ f^{(n)}(c)=n!a_n.$$

Solving for each coefficient:

$$a_0=f(c),\quad a_1=f'(c),\quad a_2=\frac{f''(c)}{2!},\quad a_3=\frac{f'''(c)}{3!},\quad a_4=\frac{f^{(4)}(c)}{4!},\quad \ldots,\ a_n=\frac{f^{(n)}(c)}{n!}$$

Taylor and Maclaurin Polynomials

With these coefficients, we can obtain the following definition of **Taylor polynomials**, named after the English mathematician Brook Taylor (1685-1731, shown at left) and **Maclaurin polynomials**, named after the Scottish mathematician Colin Maclaurin (1698-1746, shown at right).

Taylor and Maclaurin Polynomials

DEFINITIONS OF nTH TAYLOR POLYNOMIAL AND nTH MACLAURIN POLYNOMIAL

If f has n derivatives at c, then the polynomial

$$P_n(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the **nth Taylor polynomial for f at c.** If $c=0$, then

$$P_n(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n$$

is also called the **nth Maclaurin polynomial for f.**

Note: The starting index for Taylor and Maclaurin polynomials is usually $n=0$.

Example: Taylor and Maclaurin Polynomials

Find the nth-degree Maclaurin polynomial for $f(x) = e^x$. Memorize this result!

Example: Taylor and Maclaurin Polynomials

Find the nth-degree Maclaurin polynomial for $f(x) = \frac{1}{1-x}$. Memorize this result!

Example: Taylor and Maclaurin Polynomials

Find the nth-degree Maclaurin polynomial for $f(x) = \sin(x)$. Memorize this result!

Example: Taylor and Maclaurin Polynomials

Find the nth-degree Maclaurin polynomial for $f(x) = \sin(x)$. Memorize this result!

Example: Taylor and Maclaurin Polynomials

Find the nth-degree Maclaurin polynomial for $f(x) = \cos(x)$. Memorize this result!

Example: Taylor and Maclaurin Polynomials

Find the nth-degree Maclaurin polynomial for $f(x) = \cos(x)$. Memorize this result!

Example: Taylor and Maclaurin Polynomials

Find the third-degree Taylor polynomial centered at $x = \frac{\pi}{6}$ for $f(x) = \sin(x)$.

Example: Taylor and Maclaurin Polynomials

Find the fourth-degree Taylor polynomial centered at $x = 1$ for $f(x) = \ln(x)$.

Taylor and Maclaurin Polynomials

Taylor polynomials and Maclaurin polynomials can be used to approximate the value of a function at a specific point.

For instance, to approximate the value of ln(1.1), we can use Taylor polynomials for $f(x) = \ln(x)$ about $c = 1$, as shown in the next example.

There are two very important points about the accuracy of Taylor (or Maclaurin) polynomials for use in approximation.

1. The approximation is usually better for higher-degree Taylor (or Maclaurin) polynomials than for those of lower degree.
2. The approximation is usually better at x-values close to c than at x-values far from c.

Example: Taylor and Maclaurin Polynomials

Use the fourth-degree Taylor polynomial centered at $x = 1$ for $f(x) = \ln(x)$ to approximate ln(1.1).