§7.7 Indeterminate Forms and L'Hôpital's Rule

Indeterminate Forms

L'Hôpital's Rule

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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Learning Goals: Students will be able to…

- Recognize limits that produce indeterminate forms.
- Apply L'Hôpital's Rule to evaluate a limit.

Learning Objectives: Students will be able to…

1.1A Express limits symbolically using correct notation, and interpret limits expressed symbolically.

1.1C Determine limits of functions.

2.1C Calculate derivatives.

2.1D Determine higher order derivatives.

Indeterminate Forms

Recall that the forms $0/0$ and ∞/∞ are called *indeterminate* because they do not guarantee that a limit exists, nor do they indicate what the limit is if one does exist.

When we encountered one of these indeterminate forms earlier in Chapter 1, we attempted to rewrite the expression by using various algebraic techniques.

Not all indeterminate forms, however, can be evaluated by algebraic manipulation. This is often true when both algebraic and transcendental functions are involved.

Indeterminate Forms

In addition to the forms $0/0$ and ∞/∞, there are other indeterminate forms such as $0\cdot\infty$, 1^{∞}, ∞^{0}, 0^{0}, and $\infty-\infty$. We *can* attempt to convert each of these forms to $0/0$ or ∞/∞ so that we can apply the techniques we will learn in this lesson. *However, only the forms $0/0$ and ∞/∞ are covered on the AP Calculus exam.*

There are similar forms that we should recognize as "determinate."

$$\infty+\infty\to\infty \qquad -\infty-\infty\to-\infty \qquad 0^{\infty}\to 0 \qquad 0^{-\infty}\to\infty$$

L'Hôpital's Rule

The limit $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{x}$ produces the indeterminate form $\dfrac{0}{0}$.

To find this limit, we can use a theorem called **L'Hôpital's Rule**, named after the French mathematician Guillaume L'Hôpital (1661-1704).

This theorem states that under certain conditions, the limit of the quotient $\dfrac{f(x)}{g(x)}$ is determined by the limit of the quotient of the derivatives $\dfrac{f'(x)}{g'(x)}$.

L'Hôpital's Rule

THEOREM L'HÔPITAL'S RULE

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x)\neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

Note: Some textbooks also spell this "L'Hospital's Rule" (the S is silent).

L'Hôpital's Rule

L'Hôpital's Rule can also be applied to one-sided limits. For instance, if the limit of $\dfrac{f(x)}{g(x)}$ as x approaches c *from the right* produces the indeterminate form $\dfrac{0}{0}$, then $\lim\limits_{x\to c^+} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to c^+} \dfrac{f'(x)}{g'(x)}$, provided this limit exists (or is infinite).

Example: L'Hôpital's Rule

Evaluate the limit: $\lim\limits_{x\to 0} \dfrac{\arctan(x)}{x}$

Example: L'Hôpital's Rule

Evaluate the limit: $\lim\limits_{x\to \infty} \dfrac{\ln(x^4)}{x^3}$

Example: L'Hôpital's Rule

Evaluate the limit: $\lim\limits_{x\to \infty} \dfrac{e^{4x}}{x^2}$