§7.5 Partial Fractions

Partial Fractions

Linear Factors

Closing Thoughts

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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Learning Goals: Students will be able to…

- Understand the concept of partial fraction decomposition.
- Use partial fraction decomposition with distinct linear factors to integrate rational functions.

Learning Objectives: Students will be able to…

2.1C Calculate derivatives.

3.1A Recognize antiderivatives of basic functions.

3.3B Calculate antiderivatives, and evaluate definite integrals.

Partial Fractions

This section examines a procedure for decomposing a rational function into simpler rational functions to which we can apply the basic integration formulas. This procedure is called the **method of partial fractions**.

Recall from algebra that every polynomial with real coefficients can be factored into linear and irreducible quadratic factors. Using factorization, we can write the partial fraction decomposition of a given rational expression.

Partial Fractions

DECOMPOSITION OF $N(x)/D(x)$ INTO PARTIAL FRACTIONS

1. **Divide if improper:** If $N(x)/D(x)$ is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator:** Completely factor the denominator into factors of the form

$$(px+q)^m \quad \text{and} \quad (ax^2+bx+c)^n$$

where ax^2+bx+c is irreducible.

3. **Linear factors:** For each factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

Note: Only the case of distinct linear factors is covered on the AP Calculus BC exam.

Linear Factors

1. Factor the denominator.
2. Rewrite the rational expression using partial fraction decomposition.
3. Multiply this equation by the least common denominator to yield the **basic equation**.
4. Substitute any *convenient* values of x to solve for A, B, etc. The most convenient values are often the roots of the distinct linear factors in the basic equation.

Example: Linear Factors

Find the indefinite integral $\int \frac{10x-3}{2x^2-x}\,dx$.

Example: Linear Factors

Evaluate the definite integral $\int_1^2 \frac{x-15}{x^2-9}\,dx$.

Example: Linear Factors

Find the indefinite integral $\int \frac{x^3+x^2-2x-3}{x^2+x-2}\,dx$.

Example: Linear Factors

Evaluate the definite integral $\int_0^{\pi/3} \frac{\sin(x)}{\cos(x)+\cos^2(x)}\,dx$.

Closing Thoughts

Before concluding this section, here are a few things to remember.

1. It is not necessary to use the partial fraction techniques on all rational functions. For instance, the following integral is evaluated more easily by the Log Rule.

$$\int \frac{x^2+1}{x^3+3x-4}\,dx = \frac{1}{3}\int \frac{3x^2+3}{x^3+3x-4}\,dx = \frac{1}{3}\ln\left|x^3+3x-4\right|+C$$

2. When the integrand is not in reduced form, reducing it may eliminate the need for partial fractions, as shown in the following integral.

$$\int \frac{x^2-x-2}{x^3-2x-4}\,dx = \int \frac{(x+1)(x-2)}{(x-2)(x^2+2x+2)}\,dx = \int \frac{x+1}{x^2+2x+2}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx$$
$$= \frac{1}{2}\ln\left|x^2+2x+2\right|+C$$