§6.4 Arc Length

Arc Length

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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Learning Goals: Students will be able to…

- Find the arc length of a smooth curve.

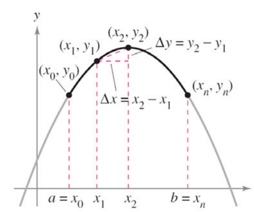
Arc Length

In this section, definite integrals are used to find the arc lengths of curves.

An arc (a segment of a curve) is approximated by straight line segments whose lengths are given by the familiar distance formula $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

A **rectifiable** curve is one that has a finite arc length. We will see that a sufficient condition for the graph of a function f to be rectifiable between $(a, f(a))$ and $(b, f(b))$ is that f' be continuous on $[a, b]$. Such a function is **continuously differentiable** on $[a, b]$, and its graph on the interval $[a, b]$ is a **smooth curve**.

Arc Length



Consider a function $y=f(x)$ that is continuously differentiable on the interval $[a, b]$. We can approximate the graph of f by n line segments whose endpoints are determined by the partition $a=x_0<x_1<x_2<\cdots<x_n=b$, as shown in the figure.

By letting $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, we can approximate the length of the graph by:

$$s \approx \sum_{i=1}^{n}\sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2}=\sum_{i=1}^{n}\sqrt{(\Delta x_i)^2+(\Delta y_i)^2}=\sum_{i=1}^{n}\sqrt{(\Delta x_i)^2+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2(\Delta x_i)^2}$$

$$=\sum_{i=1}^{n}\sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\sqrt{(\Delta x_i)^2}=\sum_{i=1}^{n}\sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$$

Arc Length

This approximation appears to become better and better as $\|\Delta\|\to 0$ $(n\to\infty)$. So, the length of the graph is $s=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$.

Because $f'(x)$ exists for each x in (x_{i-1}, x_i), the Mean Value Theorem guarantees the existence of c_i in (x_{i-1}, x_i) such that $f'(c_i)=\frac{\Delta y_i}{\Delta x_i}$.

Because f' is continuous on $[a, b]$, it follows that $\sqrt{1+[f'(x)]^2}$ is also continuous (and therefore integrable) on $[a, b]$, which implies that $s=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\sqrt{1+[f'(c_i)]^2}\,\Delta x_i \Rightarrow$

$s=\int_a^b\sqrt{1+[f'(x)]^2}\,dx$, where s is called the **arc length** of f between a and b.

Arc Length

DEFINITION OF ARC LENGTH

Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Similarly, for a smooth curve given by $x = g(y)$, the **arc length** of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Example: Arc Length

Find the arc length of the graph of $y = \ln(\sin(x))$ over the interval $[\pi/4,\ 3\pi/4]$.

Example: Arc Length

Find the arc length of the graph of $x = \frac{3}{2}y^{2/3}$ over the interval [1, 8].

Example: Arc Length

Find the arc length of the graph of $y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$ over the interval [1, 3].

Example: Arc Length (cont.)

Find the arc length of the graph of $y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$ over the interval [1, 3].

Example: Arc Length

Let R be the region bounded by the graphs of $f(x) = e^x$ and $g(x) = x + 2$. Find the perimeter of R.