§5.4 The Logistic Equation

Logistic Differential Equation

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
Calculus, AP Edition, 9th ed. by Larson & Edwards. © 2010 Brooks/Cole, Cengage Learning.

Learning Goals: Students will be able to…

- Solve and analyze logistic differential equations.
- Use logistic differential equations to model and solve applied problems.

Learning Objectives: Students will be able to…

3.5B Interpret, create, and solve differential equations from problems in context.

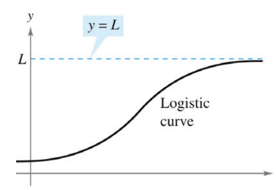
Logistic Differential Equation

The differential equation $\frac{dy}{dt} = ky$ has the general solution $y = Ce^{kt}$.

Exponential growth is unlimited, but when describing a population, there often exists some upper limit L past which growth cannot occur. This upper limit L is called the **carrying capacity**, which is the maximum population $y(t)$ that can be sustained or supported as time t increases.

Logistic Differential Equation

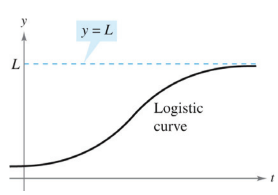
A model that is often used to describe this type of growth is the **logistic differential equation** $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$, where k and L are positive constants. A population that satisfies this equation does not grow without bound, but approaches the carrying capacity L as t increases.



Logistic Differential Equation

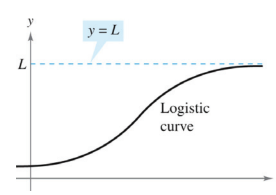
From the equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$, we can see that if y is between 0 and the carrying capacity L, then $dy/dt > 0$, and the population increases.

If y is greater than L, then $dy/dt < 0$, and the population decreases.



Logistic Differential Equation

Using separation of variables and partial fractions (a BC-only topic covered in §7.5), we can determine that all solutions of the logistic differential equation are of the general form $y = \frac{L}{1 + be^{-kt}}$, where L is still the carrying capacity, k is still the same positive constant, and b is a constant.



The graph of the function y is called the *logistic curve*.

For logistic *growth* curves, the population is growing most rapidly (that is, dy/dt has an absolute maximum value; or the graph of y has a point of inflection) at the time t that corresponds to $y = L/2$.

Example: Logistic Differential Equation

The logistic equation $P(t) = \dfrac{2000}{1 + 39e^{-0.45t}}$ models the growth of a population, where t is measured in years.

(a) Find the value of k.
(b) Find the carrying capacity.
(c) Find the initial population.

Example: Logistic Differential Equation

The logistic equation $P(t) = \dfrac{2000}{1 + 39e^{-0.45t}}$ models the growth of a population, where t is measured in years.

(d) Determine when the population will reach 50% of its carrying capacity.
(e) Write a logistic differential equation that has the solution $P(t)$.

Example: Logistic Differential Equation

The logistic differential equation $\dfrac{dy}{dt} = 0.3y - 0.0015y^2$ models the growth rate of a population, where t is measured in years.

(a) Find the value of k.
(b) Find the carrying capacity.
(c) Determine the value of y at which the population growth rate is the greatest.

Example: Logistic Differential Equation

The logistic differential equation $\dfrac{dy}{dt} = 0.3y - 0.0015y^2$ models the growth rate of a population, where t is measured in years.

(d) Find the logistic equation that satisfies the initial condition $y(0) = 10$.

Example: Logistic Differential Equation

At time $t = 0$, a bacterial culture weighs 4 grams. Four hours later, the culture weighs 12 grams. The maximum weight of the culture is 60 grams.

(a) Write a logistic equation that models the weight of the bacterial culture.

Example: Logistic Differential Equation

At time $t = 0$, a bacterial culture weighs 4 grams. Four hours later, the culture weighs 12 grams. The maximum weight of the culture is 60 grams.

(b) Find the culture's weight after seven hours.
(c) After how many hours is the culture's weight increasing most rapidly? Explain.