§5.3 Separation of Variables

Separation of Variables

Learning Goals: Students will be able to…

- Recognize and solve differential equations that can be solved by separation of variables.
- Use differential equations to model and solve applied problems.

Learning Objectives: Students will be able to…

2.1C Calculate derivatives.
3.1A Recognize antiderivatives of basic functions.
3.3B Calculate antiderivatives, and evaluate definite integrals.
3.5A Analyze differential equations to obtain general and specific solutions.
3.5B Interpret, create, and solve differential equations from problems in context.

Separation of Variables

Consider a differential equation that can be written in the form $M(x)+N(y)\frac{dy}{dx}=0$, where M is a continuous function of x alone and N is a continuous function of y alone.

For this type of equation, all x terms can be collected with dx and all y terms with dy, and a solution can be obtained via integration.

$$M(x)+N(y)\frac{dy}{dx}=0$$
$$N(y)\frac{dy}{dx}=-M(x)$$
$$\int N(y)\,dy=\int -M(x)\,dx$$

Separation of Variables

Such equations are said to be **separable**, and the solution procedure is called **separation of variables**.

Original Differential Equation	Rewritten with Variables Separated
$\frac{dy}{dx}=\frac{x}{y^2+1}$	$(y^2+1)\,dy=x\,dx$
$x^2+3y\frac{dy}{dx}=0$	$3y\,dy=-x^2\,dx$
$\sin(x)\,y'=\cos(x)$	$dy=\cot(x)\,dx$
$\frac{xy'}{e^y+1}=2$	$\frac{dy}{e^y+1}=\frac{2}{x}\,dx$

Separation of Variables

1. Perform separation of variables.
2. Perform antidifferentiation on each side.
3. Include constant of integration C on the side with the independent variable.
4. Use the initial condition to solve for C.
5. Solve for the dependent variable.
   - Rewrite any absolute values based on the initial condition.
     Replace $|a|$ with a if $a\geq 0$ for the initial condition.
     Replace $|a|$ with $-a$ if $a<0$ for the initial condition.

Example: Separation of Variables

Given the differential equation $\sqrt{x}+\sqrt{y}\frac{dy}{dx}=0$, find the particular solution that satisfies the initial condition $y(9)=1$.

Example: Separation of Variables

Given the differential equation $xy' = y^2$ find the particular solution that satisfies the initial condition $y(1) = 2$.

Example: Separation of Variables

Given the differential equation $4xy + y' = 0$, find the particular solution that satisfies the initial condition $y(2) = -1$.

Example: Separation of Variables

Given the differential equation $yy' = 2\sin(4x)$, find the particular solution that satisfies the initial condition $y(0) = -2$.