§4.7 Inverse Trigonometric Functions: Integration

Integrals Involving Inverse Trigonometric Functions

Completing the Square

Review of Basic Integration Rules

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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Learning Goals: Students will be able to…

- Integrate functions whose antiderivatives involve inverse trigonometric functions.
- Use the method of completing the square to integrate a function.
- Review the basic integration rules involving elementary functions.

Integrals Involving Inverse Trigonometric Functions

The derivatives of the six inverse trigonometric functions fall into three pairs. In each pair, the derivative of one function is the negative of the other. For example:

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}} \text{ and } \frac{d}{dx}[\arccos(x)] = -\frac{1}{\sqrt{1-x^2}}$$

When listing the *antiderivative* that corresponds to each of the inverse trigonometric functions, we need to use only one member from each pair.

It is conventional to use $\arcsin(x)$ as the antiderivative of $\frac{1}{\sqrt{1-x^2}}$, rather than $-\arccos(x)$.

Similar conventions hold true for the other two pairs, as well.

Integrals Involving Inverse Trigonometric Functions

THEOREM INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x, and let $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$

2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

Example: Integrals Involving Inverse Trigonometric Functions

Find the indefinite integral.

$$\int \frac{1}{x\sqrt{16x^2 - 9}}\,dx$$

Example: Integrals Involving Inverse Trigonometric Functions

Evaluate the definite integral.

$$\int_0^{3/4} \frac{1}{\sqrt{9 - 4x^2}}\,dx$$

Example: Integrals Involving Inverse Trigonometric Functions

Find the particular solution that satisfies the differential equation $f'(x) = \dfrac{e^x}{16 + e^{2x}}$ and the initial condition $f(\ln(4)) = \pi$.

Completing the Square

Completing the square helps when quadratic functions are involved in the integrand.

For example, the quadratic $x^2 + bx + c$ can be written as either a sum or difference of two squares by adding and subtracting $(b/2)^2$.

$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = \left(x + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2$$

Example: Completing the Square

Evaluate the definite integral.

$$\int_{-2}^{1} \frac{dx}{x^2 + 4x + 13}$$

Example: Completing the Square

Find the indefinite integral.

$$\int \frac{1}{\sqrt{-x^2 + 14x - 45}}\, dx$$

Review of Basic Integration Rules

We have now completed the introduction of the **basic integration rules**. To be efficient in applying these rules, you should have practiced enough so that each rule is committed to memory.

You are required to memorize the 20 integration rules on the next slide.

Given the left side of each equation (in random order), you must be able to write down the right side of all 20 equations in five minutes. This will constitute a **gateway exam**. You must get all 20 integration rules correct in order to pass the gateway exam.

Review of Basic Integration Rules

$\int kf(u)\, du = k\int f(u)\, du$

$\int [f(u) \pm g(u)]\, du = \int f(u)\, du \pm \int g(u)\, du$

$\int du = u + C$

$\int u^n\, du = \dfrac{u^{n+1}}{n+1} + C,\ n \neq -1$

$\int \dfrac{du}{u} = \ln|u| + C$

$\int e^u\, du = e^u + C$

$\int a^u\, du = \dfrac{a^u}{\ln(a)} + C$

$\int \sin(u)\, du = -\cos(u) + C$

$\int \cos(u)\, du = \sin(u) + C$

$\int \tan(u)\, du = -\ln|\cos(u)| + C$

$\int \cot(u)\, du = \ln|\sin(u)| + C$

$\int \sec(u)\, du = \ln|\sec(u) + \tan(u)| + C$

$\int \csc(u)\, du = -\ln|\csc(u) + \cot(u)| + C$

$\int \sec^2(u)\, du = \tan(u) + C$

$\int \csc^2(u)\, du = -\cot(u) + C$

$\int \sec(u)\tan(u)\, du = \sec(u) + C$

$\int \csc(u)\cot(u)\, du = -\csc(u) + C$

$\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\dfrac{u}{a}\right) + C$

$\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\arctan\left(\dfrac{u}{a}\right) + C$

$\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\operatorname{arcsec}\dfrac{|u|}{a} + C$