§4.6 The Natural Logarithmic Function: Integration

Log Rule for Integration

Integrals of Trigonometric Functions

Learning Goals: Students will be able to…

- Use the Log Rule for Integration to integrate a rational function.
- Integrate trigonometric functions.

Learning Objectives: Students will be able to…

2.1C Calculate derivatives.
3.1A Recognize antiderivatives of basic functions.
3.3B Calculate antiderivatives, and evaluate definite integrals.
3.4B Apply definite integrals to problems involving the average value of a function.
3.5A Analyze differential equations to obtain general and specific solutions.

Log Rule for Integration

THEOREM LOG RULE FOR INTEGRATION

Let u be a differentiable function of x.

1. $\int \frac{1}{x}\,dx = \ln|x| + C$ **2.** $\int \frac{1}{u}\,du = \ln|u| + C$

In Chapter 2, we studied two differentiation rules for logarithms.

The differentiation rule $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$ produces the integration rule that we learned in §4.1.

The differentiation rule $\frac{d}{dx}[\ln(u)] = \frac{u'}{u}$ produces the new integration rule in this lesson.

Example: Log Rule for Integration

Find the indefinite integral.

$$\int \frac{x}{x^2 - 3}\,dx$$

Example: Log Rule for Integration

Find the average value of $f(x) = \frac{[1+\ln(x)]^2}{x}$ on the interval $[1, e]$.

Integrals of Trigonometric Functions

In §4.1, we looked at six trigonometric integration rules—the six that correspond directly to differentiation rules. With the Log Rule, we can now complete the set of basic trigonometric integration formulas.

Integrals of Trigonometric Functions

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{1}{\cos(x)}\sin(x)\,dx = -\int \frac{1}{u}\,du$$
$$= -\ln|u| + C = -\ln|\cos(x)| + C$$

Let $u = \cos(x)$. Then $du = -\sin(x)\,dx \Rightarrow -du = \sin(x)\,dx$.

Integrals of Trigonometric Functions

$$\int \sec(x)\,dx = \int \sec(x)\cdot\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$$
$$= \int \frac{1}{\sec(x)+\tan(x)}\left[\sec^2(x)+\sec(x)\tan(x)\right]dx = \int \frac{1}{u}\,du$$
$$= \ln|u| + C = \ln|\sec(x)+\tan(x)| + C$$

Let $u = \sec(x)+\tan(x)$. Then $du = \left[\sec(x)\tan(x)+\sec^2(x)\right]dx$.

Integrals of Trigonometric Functions

INTEGRALS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$
$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$
$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

Note: Some textbooks or websites may have different, but equivalent, rules for $\int \sec(u)\,du$ and $\int \csc(u)\,du$.

Example: Integrals of Trigonometric Functions

Find the indefinite integral.

$$\int \left[2 - \tan\left(\frac{\theta}{4}\right)\right] d\theta$$

Example: Integrals of Trigonometric Functions

Find the particular solution that satisfies the differential equation $f'(x) = 6\csc(3x)$ and the initial condition $f(\pi/6) = 4$.