§4.5 Integration by Substitution

Pattern Recognition

Change of Variables for Indefinite Integrals

Change of Variables for Definite Integrals

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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Learning Goals: Students will be able to…

- Use pattern recognition to find an indefinite integral.
- Use a change of variables to find an indefinite integral.
- Use a change of variables to evaluate a definite integral.

Pattern Recognition

In this section, we will study techniques for integrating composite functions. Our techniques involve a ***u*-substitution.** The role of substitution in integration is comparable to the role of the Chain Rule in differentiation.

Recall that for the differentiable functions $y = F(u)$ and $u = g(x)$ the Chain Rule states that $\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))g'(x)$.

From the definition of an antiderivative, it follows that $\int F'(g(x))g'(x)\,dx = F(g(x)) + C$.

Pattern Recognition

THEOREM ANTIDIFFERENTIATION OF A COMPOSITE FUNCTION

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

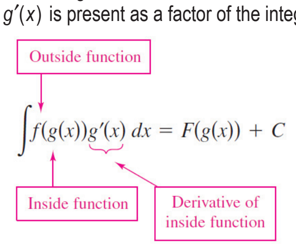
$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

Pattern Recognition

The following examples will show how to recognize the presence of $f(g(x))$ and $g'(x)$. Note that the composite function in the integrand has an *outside function* f and an *inside function* g. Moreover, the derivative $g'(x)$ is present as a factor of the integrand.



Example: Pattern Recognition

For the given indefinite integral of the form $\int f(g(x))g'(x)\,dx$, identify $u = g(x)$ and $du = g'(x)\,dx$.

$\int (8x^2+1)^2(16x)\,dx$

Example: Pattern Recognition

For the given indefinite integral of the form $\int f(g(x))g'(x)\,dx$, identify $u=g(x)$ and $du=g'(x)\,dx$.

$\int \tan^2(x)\sec^2(x)\,dx$

Pattern Recognition

The integrands in the previous examples fit the $f(g(x))g'(x)$ pattern exactly—we only had to recognize the pattern. We can extend this technique considerably with the Constant Multiple Rule $\int kf(x)\,dx = k\int f(x)\,dx$.

Many integrands contain the essential part (the variable part) of $g'(x)$ but are missing a constant multiple. In such cases, we can multiply and divide by the necessary constant multiple.

Example: Pattern Recognition

For the given indefinite integral of the form $\int f(g(x))g'(x)\,dx$, identify $u=g(x)$ and $du=g'(x)\,dx$.

$\int x^2(x^3+1)^5\,dx$

Pattern Recognition

Be sure to see that the *Constant* Multiple Rule applies only to *constants*. We cannot multiply and divide by a variable and then move the variable outside the integral sign.

$\int (x^3+1)^5\,dx = \int (x^3+1)^5\,dx \neq \frac{1}{3x^2}\int u^5\,du$

Let $u=x^3+1$. Then $du=3x^2\,dx \Rightarrow \frac{1}{3x^2}\,du = dx$.

After all, if it were legitimate to move variable quantities outside the integral sign, we could move the entire integrand out and simplify the whole process. But the result would be incorrect.

Change of Variables for Indefinite Integrals

With a formal **change of variables**, we completely rewrite the integral in terms of u and du (or any other convenient variable). Although this procedure can involve more written steps than is necessary for simpler problems, it is useful for more complicated integrands.

The change in variables technique uses the Leibniz notation for the differential. That is, if $u=g(x)$, then $du=g'(x)\,dx$, and the integral takes the form

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C.$$

Change of Variables for Indefinite Integrals

GUIDELINES FOR MAKING A CHANGE OF VARIABLES

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\,dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

Example: Change of Variables for Indefinite Integrals

Find the indefinite integral.

$\int x(5x^2+4)^3\,dx$

Example: Change of Variables for Indefinite Integrals

Find the indefinite integral.

$\int x\csc^2(x^2)\,dx$

Example: Change of Variables for Indefinite Integrals

Find the particular solution to the differential equation $f'(x)=\dfrac{\sin(x)}{\cos^3(x)}$ with initial condition $f(0)=3$.

Change of Variables for Definite Integrals

THEOREM CHANGE OF VARIABLES FOR DEFINITE INTEGRALS

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

When using u-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable u rather than convert the antiderivative back to the variable x and evaluate at the original limits. This change of variables is stated explicitly in the above theorem.

Example: Change of Variables for Definite Integrals

Evaluate the definite integral.

$\int_0^2 \dfrac{x}{\sqrt{1+2x^2}}\,dx$

Example: Change of Variables for Definite Integrals

Find the average value of $f(x)=e^{-2x}$ on the interval [0, 2].