§4.4a The First Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

Average Value of a Function

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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Learning Goals: Students will be able to…

- Evaluate a definite integral using the Fundamental Theorem of Calculus.
- Find the average value of a function over a closed interval.

The Fundamental Theorem of Calculus

We have now been introduced to the two major branches of calculus: differential calculus (introduced with the tangent line problem) and integral calculus (introduced with the area problem).

So far, these two problems might seem unrelated—but there is a very close connection. The connection was discovered independently by Isaac Newton and Gottfried Leibniz and is stated in the **Fundamental Theorem of Calculus**.

Informally, the theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations.

The Fundamental Theorem of Calculus

Throughout this chapter, we have been using the integral sign to denote an antiderivative (a family of functions) and a definite integral (a number).

Antidifferentiation: $\int f(x)\,dx$ Definite integration: $\int_a^b f(x)\,dx$

The use of the same symbol for both operations makes it appear that they are related.

The Fundamental Theorem of Calculus

THEOREM THE FUNDAMENTAL THEOREM OF CALCULUS

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

When applying the Fundamental Theorem of Calculus, the notation shown below is convenient.

$$\int_a^b f(x)\,dx = \left[F(x)\right]_{x=a}^{x=b} = F(b) - F(a)$$

The Fundamental Theorem of Calculus

THEOREM THE FUNDAMENTAL THEOREM OF CALCULUS

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

It is not necessary to include a constant of integration C in the antiderivative.

$$\int_a^b f(x)\,dx = \left[F(x)+C\right]_{x=a}^{x=b} = \left[F(b)+C\right]-\left[F(a)+C\right]$$
$$= F(b)+C-F(a)-C = F(b)-F(a)$$

Example: The Fundamental Theorem of Calculus

Evaluate the definite integral:

$\int_1^7 (6x^2 + 2x - 3)\, dx$

Example: The Fundamental Theorem of Calculus

Evaluate the definite integral:

$\int_e^{2e} \left(\cos(x) - \frac{1}{x} \right) dx$

Average Value of a Function

DEFINITION OF THE AVERAGE VALUE OF A FUNCTION ON AN INTERVAL

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\, dx.$$

Example: Average Value of a Function

Find the average value of the function $f(x) = \cos(x)$ over the interval $[0, \pi/2]$.

Example: Average Value of a Function

A particle moves along the x-axis with velocity given by the function $v(t) = t + e^t$. Find the average velocity of the particle from time $t = 0$ to $t = 3$.

Average Value of a Function

Suppose $s(t)$ represents a particle's position at time t, $v(t)$ represents the particle's velocity at time t, and $a(t)$ represents the particle's acceleration at time t.

Average velocity on $[t_1, t_2]$: $\frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$ $\qquad \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)\, dt$

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)\, dt = \frac{1}{t_2 - t_1}\left[s(t)\right]_{t=t_1}^{t=t_2}$$

$$= \frac{1}{t_2 - t_1}\left[s(t_2) - s(t_1)\right]$$

$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

Average Value of a Function

Suppose $s(t)$ represents a particle's position at time t, $v(t)$ represents the particle's velocity at time t, and $a(t)$ represents the particle's acceleration at time t.

Average acceleration on $[t_1, t_2]$: $\dfrac{\Delta v}{\Delta t} = \dfrac{v(t_2) - v(t_1)}{t_2 - t_1}$ $\qquad \dfrac{1}{t_2 - t_1}\int_{t_1}^{t_2} a(t)\, dt$

$$\begin{aligned} \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} a(t)\, dt &= \frac{1}{t_2 - t_1}\left[v(t)\right]_{t=t_1}^{t=t_2} \\ &= \frac{1}{t_2 - t_1}\left[v(t_2) - v(t_1)\right] \\ &= \frac{v(t_2) - v(t_1)}{t_2 - t_1} \end{aligned}$$