§4.1 Antiderivatives and Indefinite Integration

Antiderivatives

Basic Integration Rules

Initial Conditions and Particular Solutions

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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Learning Goals: Students will be able to…

- Write the general solution of a differential equation.
- Use indefinite integral notation for antiderivatives.
- Use basic integration rules to find antiderivatives.
- Find a particular solution of a differential equation.

Antiderivatives

DEFINITION OF ANTIDERIVATIVE

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

If $f(x) = 3x^2$, then we could say that $F(x) = x^3$.

Confirmation: $F'(x) = 3x^2 = f(x)$

Therefore, x^3 is an antiderivative of $3x^2$.

But is x^3 the only antiderivative of $3x^2$?

Nope! Consider $x^3 - 5$, or $x^3 + 97$. These are also antiderivatives of $3x^2$!

Note that all three of these antiderivatives differ by a *constant* value.

Antiderivatives

THEOREM REPRESENTATION OF ANTIDERIVATIVES

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$, for all x in I where C is a constant.

Antiderivatives

Using the aforementioned theorem, we can represent the entire family of antiderivatives of a function by adding a constant to a *known* derivative.

For example, knowing that $\frac{d}{dx}\left[x^2\right] = 2x$, we can represent the family of *all* antiderivatives of $f(x) = 2x$ by $G(x) = x^2 + C$, where C is a constant.

The constant C is called the **constant of integration**. The family of functions represented by G is the **general antiderivative** of f, and $G(x) = x^2 + C$ is the **general solution** of the *differential equation* $G'(x) = 2x$.

A **differential equation** in x and y is an equation that involves x, y, and derivatives of y.

For example: $y' = 3x$ and $y' = x^2 + 1$ are differential equations.

Example: Antiderivatives

Find the general solution of the differential equation $y' = 2$. Check the result by differentiation.

Antiderivatives

Some simple differential equations are of the form $\frac{dy}{dx} = f(x)$.

When solving a differential equation of this form, it is convenient to write it in the equivalent **differential** form, $dy = f(x)\,dx$.

The operation of finding all solutions of an equation $dy = f(x)\,dx$ is called **antidifferentiation** (or **indefinite integration**) and is denoted by an integral sign: $\int$

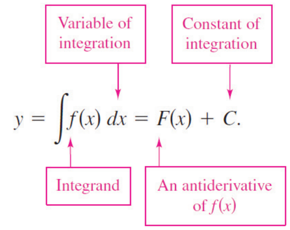
The general solution is denoted by: $y = \int f(x)\,dx = F(x) + C$

Antiderivatives

The expression $\int f(x)\,dx$ is read as "the antiderivative of f with respect to x."

The *differential* dx serves to identify x as the variable of integration.

The term **indefinite integral** is a synonym for antiderivative.



Basic Integration Rules

The inverse nature of integration and differentiation can be verified by substituting $F'(x)$ for $f(x)$ in the indefinite integration definition to obtain:

$$\int F'(x)\,dx = F(x) + C.$$

Integration is the "inverse" of differentiation.

Moreover, if $\int f(x)\,dx = F(x) + C$, then:

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

Differentiation is the "inverse" of integration.

These two equations allow us to obtain integration formulas directly from differentiation formulas.

Basic Integration Rules

Differentiation Formula	*Integration Formula*
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$

Basic Integration Rules

Differentiation Formula	*Integration Formula*
$\frac{d}{dx}[\sin(x)] = \cos(x)$	$\int \cos(x)\,dx = \sin(x) + C$
$\frac{d}{dx}[\cos(x)] = -\sin(x)$	$\int \sin(x)\,dx = -\cos(x) + C$
$\frac{d}{dx}[\tan(x)] = \sec^2(x)$	$\int \sec^2(x)\,dx = \tan(x) + C$
$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$	$\int \sec(x)\tan(x)\,dx = \sec(x) + C$
$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$	$\int \csc^2(x)\,dx = -\cot(x) + C$
$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$	$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$

Basic Integration Rules

Differentiation Formula	*Integration Formula*
$\frac{d}{dx}\left[e^x\right] = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}\left[a^x\right] = a^x\ln(a)$	$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$
$\frac{d}{dx}[\ln(x)] = \frac{1}{x},\ x > 0$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

Example: Basic Integration Rules

Find the indefinite integral $\int (1+3t)t^2\, dt$ and check the result by differentiation.

Example: Basic Integration Rules

Find the indefinite integral $\int \frac{x^3+2x-3}{x^4}\, dx$ and check the result by differentiation.

Example: Basic Integration Rules

Find the indefinite integral $\int \frac{\sin(x)}{1-\sin^2(x)}\, dx$ and check the result by differentiation.

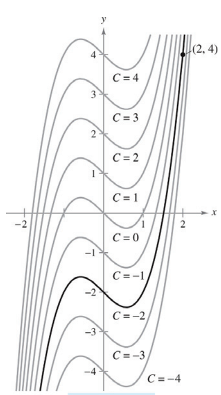
Example: Basic Integration Rules

Find the indefinite integral $\int (5\csc^2(x) - 7e^x)\, dx$ and check the result by differentiation.

Initial Conditions and Particular Solutions

We have already seen that the equation $y = \int f(x)\, dx$ has many solutions, each differing from the others by a constant. This means that the graphs of any two antiderivatives of f are vertical translations of each other.

For example, the figure shows the graphs of several antiderivatives of the form $y = \int (3x^2 - 1)\, dx = x^3 - x + C$ for various integer values of C. Each of these antiderivatives is a solution of the differential equation $\frac{dy}{dx} = 3x^2 - 1$.



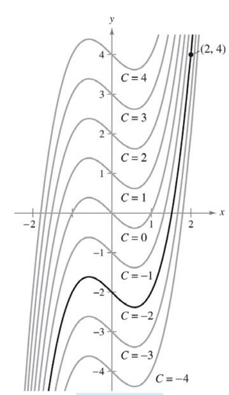
$F(x) = x^3 - x + C$

Initial Conditions and Particular Solutions

In many applications of integration, we are given enough information to determine a **particular solution**. To do this, we need only know the value of $y = F(x)$ for one value of x. This information is called an **initial condition**.

For example, in the figure, only one curve passes through the point (2, 4). To find this curve, we can use the general solution $F(x) = x^3 - x + C$ and the initial condition $F(2) = 4$.

By using the initial condition in the general solution, we can determine that $C = -2$. So we obtain $F(x) = x^3 - x - 2$.



$F(x) = x^3 - x + C$

Example: Initial Conditions and Particular Solutions

Find the particular solution that satisfies the differential equation $f''(x) = 6x - 2$ and the initial condition $f(3) = 2$ and $f'(2) = 5$.

Example: Initial Conditions and Particular Solutions

A particle, initially at rest, moves along the y-axis such that its acceleration at time $t \geq 0$ is given by $a(t) = 4\sin(t)$. At the time $t = 0$, its position is $y = 6$. Find the velocity and position functions for the particle.