§4.1 Antiderivatives and Indefinite Integration

Antiderivatives

Basic Integration Rules

Initial Conditions and Particular Solutions

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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Learning Goals: Students will be able to…

- Write the general solution of a differential equation.
- Use indefinite integral notation for antiderivatives.
- Use basic integration rules to find antiderivatives.
- Find a particular solution of a differential equation.

Learning Objectives: Students will be able to…

2.3E Verify solutions to differential equations.
3.1A Recognize antiderivatives of basic functions.
3.3B Calculate antiderivatives, and evaluate definite integrals.
3.5A Analyze differential equations to obtain general and specific solutions.

Antiderivatives

DEFINITION OF ANTIDERIVATIVE

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

If $f(x) = 3x^2$, then we could say that $F(x) = x^3$.

Confirmation: $F'(x) = 3x^2 = f(x)$

Therefore, x^3 is an antiderivative of $3x^2$.

But is x^3 the only antiderivative of $3x^2$?

Nope! Consider $x^3 - 5$, or $x^3 + 97$. These are also antiderivatives of $3x^2$!

Note that all three of these antiderivatives differ by a *constant* value.

Antiderivatives

THEOREM REPRESENTATION OF ANTIDERIVATIVES

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$, for all x in I where C is a constant.

Antiderivatives

Using the aforementioned theorem, we can represent the entire family of antiderivatives of a function by adding a constant to a *known* derivative.

For example, knowing that $\frac{d}{dx}\left[x^2\right] = 2x$, we can represent the family of *all* antiderivatives of $f(x) = 2x$ by $G(x) = x^2 + C$, where C is a constant.

The constant C is called the **constant of integration**. The family of functions represented by G is the **general antiderivative** of f, and $G(x) = x^2 + C$ is the **general solution** of the *differential equation* $G'(x) = 2x$.

A **differential equation** in x and y is an equation that involves x, y, and derivatives of y.

For example: $y' = 3x$ and $y' = x^2 + 1$ are differential equations.

Example: Antiderivatives

Find the general solution of the differential equation $y' = 2$. Check the result by differentiation.

Antiderivatives

Some simple differential equations are of the form $\frac{dy}{dx} = f(x)$.

When solving a differential equation of this form, it is convenient to write it in the equivalent **differential** form, $dy = f(x)\,dx$.

The operation of finding all solutions of an equation $dy = f(x)\,dx$ is called **antidifferentiation** (or **indefinite integration**) and is denoted by an integral sign: $\int$

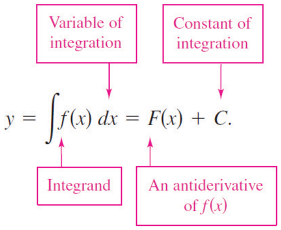
The general solution is denoted by: $y = \int f(x)\,dx = F(x) + C$

Antiderivatives

The expression $\int f(x)\,dx$ is read as "the antiderivative of f with respect to x."

The *differential* dx serves to identify x as the variable of integration.

The term **indefinite integral** is a synonym for antiderivative.



Basic Integration Rules

The inverse nature of integration and differentiation can be verified by substituting $F'(x)$ for $f(x)$ in the indefinite integration definition to obtain:

$$\int F'(x)\,dx = F(x) + C.$$

Integration is the "inverse" of differentiation.

Moreover, if $\int f(x)\,dx = F(x) + C$, then:

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

Differentiation is the "inverse" of integration.

These two equations allow us to obtain integration formulas directly from differentiation formulas.

Basic Integration Rules

Differentiation Formula	*Integration Formula*
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$

Basic Integration Rules

Differentiation Formula	*Integration Formula*
$\frac{d}{dx}[\sin(x)] = \cos(x)$	$\int \cos(x)\,dx = \sin(x) + C$
$\frac{d}{dx}[\cos(x)] = -\sin(x)$	$\int \sin(x)\,dx = -\cos(x) + C$
$\frac{d}{dx}[\tan(x)] = \sec^2(x)$	$\int \sec^2(x)\,dx = \tan(x) + C$
$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$	$\int \sec(x)\tan(x)\,dx = \sec(x) + C$
$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$	$\int \csc^2(x)\,dx = -\cot(x) + C$
$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$	$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$

Basic Integration Rules

Differentiation Formula	*Integration Formula*
$\frac{d}{dx}\left[e^x\right] = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}\left[a^x\right] = a^x \ln(a)$	$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$
$\frac{d}{dx}[\ln(x)] = \frac{1}{x},\ x > 0$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

Example: Basic Integration Rules

Find the indefinite integral $\int (1+3t)t^2 \, dt$ and check the result by differentiation.

Example: Basic Integration Rules

Find the indefinite integral $\int \frac{x^3+2x-3}{x^4} \, dx$ and check the result by differentiation.

Example: Basic Integration Rules

Find the indefinite integral $\int \frac{\sin(x)}{1-\sin^2(x)} \, dx$ and check the result by differentiation.

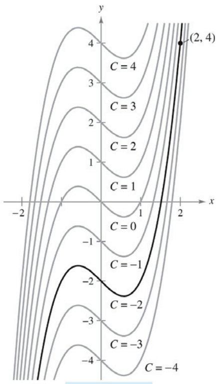
Example: Basic Integration Rules

Find the indefinite integral $\int (5\csc^2(x) - 7e^x) \, dx$ and check the result by differentiation.

Initial Conditions and Particular Solutions

We have already seen that the equation $y = \int f(x) \, dx$ has many solutions, each differing from the others by a constant. This means that the graphs of any two antiderivatives of f are vertical translations of each other.

For example, the figure shows the graphs of several antiderivatives of the form $y = \int (3x^2 - 1) \, dx = x^3 - x + C$ for various integer values of C. Each of these antiderivatives is a solution of the differential equation $\frac{dy}{dx} = 3x^2 - 1$.



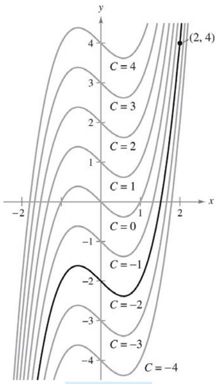
$F(x) = x^3 - x + C$

Initial Conditions and Particular Solutions

In many applications of integration, we are given enough information to determine a **particular solution**. To do this, we need only know the value of $y = F(x)$ for one value of x. This information is called an **initial condition**.

For example, in the figure, only one curve passes through the point (2, 4). To find this curve, we can use the general solution $F(x) = x^3 - x + C$ and the initial condition $F(2) = 4$.

By using the initial condition in the general solution, we can determine that $C = -2$. So we obtain $F(x) = x^3 - x - 2$.



$F(x) = x^3 - x + C$

Example: Initial Conditions and Particular Solutions

Find the particular solution that satisfies the differential equation $f''(x)=6x-2$ and the initial condition $f(3)=2$ and $f'(2)=5$.

Example: Initial Conditions and Particular Solutions

A particle, initially at rest, moves along the y-axis such that its acceleration at time $t \geq 0$ is given by $a(t)=4\sin(t)$. At the time $t=0$, its position is $y=6$. Find the velocity and position functions for the particle.