§2.6 Derivatives of Inverse Functions

Derivative of an Inverse Function

Derivatives of Inverse Trigonometric Functions

Basic Differentiation Rules for Elementary Functions

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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Learning Goals: Students will be able to…

- Find the derivative of an inverse function.
- Differentiate an inverse trigonometric function.

Derivative of an Inverse Function

THEOREM THE DERIVATIVE OF AN INVERSE FUNCTION

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

Alternatively: $(f^{-1})'(x) = \dfrac{1}{f'\left(f^{-1}(x)\right)}$, $f'\left(f^{-1}(x)\right) \neq 0$

Example: Derivative of an Inverse Function

Given the function $f(x) = x^3 - 1$, find $(f^{-1})'(26)$.

Example: Derivative of an Inverse Function

Given the function $f(x) = \dfrac{x+3}{x+1}$ for $x > -1$, find $(f^{-1})'(2)$.

Example: Derivative of an Inverse Function

Let f be a differentiable function such that $f(3) = 0$, $f(15/2) = 3$, $f'(3) = -1/3$, and $f'(15/2) = -4/3$. Find $(f^{-1})'(3)$.

Derivatives of Inverse Trigonometric Functions

THEOREM DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Example: Derivatives of Inverse Trigonometric Functions

Find the second derivative of the function $y = \arctan(2x)$.

Example: Derivatives of Inverse Trigonometric Functions

Find an equation of the line tangent to the graph of the equation $\arcsin(4x) = 3\operatorname{arcsec}(x+y) - \pi$ at the point $(0, 2)$.

Basic Differentiation Rules for Elementary Functions

As mathematics has developed during the past few hundred years, a small number of elementary functions has proven sufficient for modeling most phenomena in physics, chemistry, biology, engineering, economics, and a variety of other fields.

An **elementary function** is a function from the following list or one that can be formed as the sum, product, quotient, or composition of functions in the list.

Algebraic Functions	*Transcendental Functions*
Polynomial functions	Logarithmic functions
Rational functions	Exponential functions
Functions involving radicals	Trigonometric functions
	Inverse trigonometric functions

Basic Differentiation Rules for Elementary Functions

With the differentiation rules introduced, we can differentiate any elementary function.

You are required to memorize the 24 differentiation rules on the next slide.

Given the left side of each equation (in random order), you must be able to write down the right side of all 24 equations in five minutes. This will constitute a **gateway exam**. You must get all 24 differentiation rules correct in order to pass the gateway exam.

Basic Differentiation Rules for Elementary Functions

$\frac{d}{dx}[cu] = cu'$ $\qquad \frac{d}{dx}[u \pm v] = u' \pm v'$ $\qquad \frac{d}{dx}[uv] = uv' + u'v$

$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$ $\qquad \frac{d}{dx}[c] = 0$ $\qquad \frac{d}{dx}[u^n] = nu^{n-1}u'$

$\frac{d}{dx}[x] = 1$ $\qquad \frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$ $\qquad \frac{d}{dx}[\ln(u)] = \frac{u'}{u}$

$\frac{d}{dx}\left[e^u\right] = e^u u'$ $\qquad \frac{d}{dx}[\log_a(u)] = \frac{u'}{u\ln(a)}$ $\qquad \frac{d}{dx}\left[a^u\right] = \ln(a)\cdot a^u u'$

$\frac{d}{dx}[\sin(u)] = \cos(u)\,u'$ $\qquad \frac{d}{dx}[\cos(u)] = -\sin(u)\,u'$ $\qquad \frac{d}{dx}[\tan(u)] = \sec^2(u)\,u'$

$\frac{d}{dx}[\cot(u)] = -\csc^2(u)\,u'$ $\qquad \frac{d}{dx}[\sec(u)] = \sec(u)\tan(u)\,u'$ $\qquad \frac{d}{dx}[\csc(u)] = -\csc(u)\cot(u)\,u'$

$\frac{d}{dx}[\arcsin(u)] = \frac{u'}{\sqrt{1-u^2}}$ $\qquad \frac{d}{dx}[\arccos(u)] = \frac{-u'}{\sqrt{1-u^2}}$ $\qquad \frac{d}{dx}[\arctan(u)] = \frac{u'}{1+u^2}$

$\frac{d}{dx}[\operatorname{arccot}(u)] = \frac{-u'}{1+u^2}$ $\qquad \frac{d}{dx}[\operatorname{arcsec}(u)] = \frac{u'}{|u|\sqrt{u^2-1}}$ $\qquad \frac{d}{dx}[\operatorname{arccsc}(u)] = \frac{-u'}{|u|\sqrt{u^2-1}}$