§2.3 Product and Quotient Rules and Higher-Order Derivatives

The Product Rule
The Quotient Rule
Derivatives of Trigonometric Functions
The Tangent Line Problem
Higher-Order Derivatives

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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Learning Goals: Students will be able to…

- Find the derivatives of functions using the Product and Quotient Rules.
- Find the derivative of a trigonometric function.
- Find a higher-order derivative of a function.
- Find average and instantaneous acceleration.
- Determine whether a particle is speeding up or slowing down at a particular time.

Learning Objectives: Students will be able to…

2.1A Identify the derivative of a function as the limit of a difference quotient.
2.1B Estimate derivatives.
2.1C Calculate derivatives.
2.1D Determine higher order derivatives.
2.3B Solve problems involving the slope of a tangent line.
2.3C Solve problems involving related rates, optimization, rectilinear motion, *(BC: and planar motion)*.
2.3D Solve problems involving rates of change in applied contexts.

The Product Rule

THEOREM THE PRODUCT RULE

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Alternate version of the Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$

Example: The Product Rule

$$\frac{d}{dx}\left[(x^2+3x+1)(4x-3)\right]$$

$$\frac{d}{dx}\left[x^3\cos(x)\right]$$

The Quotient Rule

THEOREM THE QUOTIENT RULE

The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

Alternate version of the Quotient Rule: $\frac{d}{dx}\left[\frac{\text{hi}}{\text{lo}}\right] = \frac{\text{lo}\cdot\text{dhi} - \text{hi}\cdot\text{dlo}}{\text{lo}^2}$

Example: The Quotient Rule

$$\frac{d}{dx}\left[\frac{x^3-4x+1}{3x+2}\right]$$

Derivatives of Trigonometric Functions

THEOREM DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Proof: $\frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{\cos(x)\cos(x) - \sin(x)\cdot -\sin(x)}{\cos^2(x)}$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

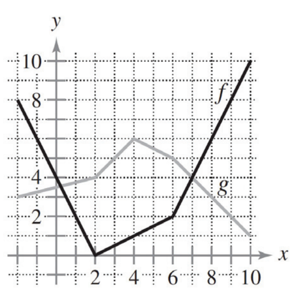
Example: Derivatives of Trigonometric Functions

$$\frac{d}{dx}\left[x^2 \cot(x)\right]$$

$$\frac{d}{dx}\left[\frac{\sec(x)}{x^3}\right]$$

Example: The Tangent Line Problem

Given the graphs of f and g, let $p(x) = f(x)g(x)$. Find an equation of the line tangent to the graph of $p(x)$ at $x = 9$.



Higher-Order Derivatives

First derivative:	y',	$f'(x)$,	$\frac{dy}{dx}$,	$\frac{d}{dx}[f(x)]$,	$D_x[y]$
Second derivative:	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$\frac{d^2}{dx^2}[f(x)]$,	$D_x^2[y]$
Third derivative:	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$\frac{d^3}{dx^3}[f(x)]$,	$D_x^3[y]$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	$\frac{d^4}{dx^4}[f(x)]$,	$D_x^4[y]$
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nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	$\frac{d^n}{dx^n}[f(x)]$,	$D_x^n[y]$

Example: Higher-Order Derivatives

Find the second derivative of $y = \csc(x) + \tan(x)$.

Higher-Order Derivatives

If $s(t)$ represents the position of an object at time t:

Velocity function	$v(t) = s'(t)$
Speed function	$\lvert v(t)\rvert = \lvert s'(t)\rvert$
Acceleration function	$a(t) = v'(t) = s''(t)$
Average velocity on $[a, b]$	$\frac{s(b) - s(a)}{b - a}$
Average acceleration on $[a, b]$	$\frac{v(b) - v(a)}{b - a}$

Higher-Order Derivatives

$v < 0 \Leftarrow$ object moves to the left $a > 0 \Rightarrow$ velocity is increasing object slows down (like pressing the brake pedal while driving in reverse)	$v > 0 \Rightarrow$ object moves to the right $a > 0 \Rightarrow$ velocity is increasing object speeds up (like pressing the gas pedal while driving forward)
$v < 0 \Leftarrow$ object moves to the left $a < 0 \Leftarrow$ velocity is decreasing object speeds up (like pressing the gas pedal while driving in reverse)	$v > 0 \Rightarrow$ object moves to the right $a < 0 \Leftarrow$ velocity is decreasing object slows down (like pressing the brake pedal while driving forward)

Example: Higher-Order Derivatives

A particle moves on the y-axis with position function given by $y(t) = -t^2 + 6t + 10$, where t is measured in seconds and $y(t)$ is measured in inches.

(a) Find the speed of the particle at time $t = 6$. Include units in your answer.
(b) Find the average acceleration of the particle over the interval [0, 3]. Include units in your answer.

Example: Higher-Order Derivatives

A particle moves on the x-axis with position function given by $x(t) = 2e^t - te^t$. Is the particle speeding up or slowing down at time $t = 3$? Explain your reasoning.