§2.2 Basic Differentiation Rules and Rates of Change

The Constant Rule
The Power Rule
The Constant Multiple Rule
The Sum and Difference Rules
Derivatives of the Sine and Cosine Functions
Derivatives of Exponential Functions
The Tangent Line Problem
Rates of Change

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Learning Goals: Students will be able to…

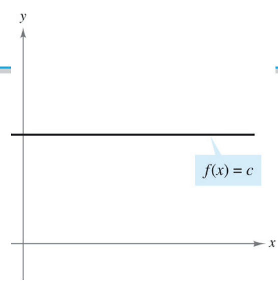
- Find the derivatives of functions using the Constant, Power, Constant Multiple, Sum, and Difference Rules.
- Find the derivatives of the sine function and of the cosine function.
- Find the derivatives of exponential functions.
- Use difference quotients to find average rates of change (including average velocity).
- Use derivatives to find instantaneous rates of change (including instantaneous velocity).
- Determine the times when a particle is at rest or changes direction.

The Constant Rule

THEOREM THE CONSTANT RULE

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$



The Power Rule

THEOREM THE POWER RULE

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.

Example: The Power Rule

$$\frac{d}{dx}[x]$$

$$\frac{d}{dx}\left[\sqrt[3]{x}\right]$$

$$\frac{d}{dx}\left[\frac{1}{x^3}\right]$$

The Constant Multiple Rule

THEOREM THE CONSTANT MULTIPLE RULE

If f is a differentiable function and c is a real number, then cf is also differentiable and $\frac{d}{dx}[cf(x)] = cf'(x)$.

Example: $\frac{d}{dx}\left[5x^2\right]$

The Sum and Difference Rules

THEOREM THE SUM AND DIFFERENCE RULES

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \qquad \text{Difference Rule}$$

Example: $\frac{d}{dx}\left[4x^5 + 2x^{-3} - 5x^{-1}\right]$

Derivatives of the Sine and Cosine Functions

THEOREM DERIVATIVES OF SINE AND COSINE FUNCTIONS

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

Example: $\frac{d}{dx}[4\sin(x) - 5\cos(x)]$

Derivatives of Exponential Functions

THEOREM DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

$$\frac{d}{dx}[e^x] = e^x$$

Example: $\frac{d}{dx}\left[4e^x\right]$

Example: The Tangent Line Problem

Find an equation of the line tangent to the graph of $f(x) = 6x^3 - 4\sin(x) + 7e^x$ at $x = 0$.

Rates of Change

You have seen how the derivative is used to determine slope.

The derivative can also be used to determine the rate of change of one variable with respect to another.

Applications involving rates of change occur in a wide variety of fields. A few examples are population growth rates, production rates, water flow rates, velocity, and acceleration.

Rates of Change

There are two types of rates of change.

The **average rate** of change of a function f with respect to x is always on a closed interval $[a, b]$, and can be calculated as follows:

$$\frac{f(b) - f(a)}{b - a}$$

The **instantaneous rate** of change of a function f with respect to x is always at one specific x-value, and is simply $f'(x)$ evaluated at that x-value.

Example: Rates of Change

Given the function $f(t) = t^3 - 7$:

(a) Find the average rate of change of f over the interval [3, 3.1].

(b) Find the instantaneous rate of change of f at $t = 3$ and $t = 3.1$.

Rates of Change

A common use for rate of change is to describe the motion of an object moving in a straight line.

In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion.

On such lines, movement to the right (or upward) is considered to be in the positive direction. Movement to the left (or downward) is considered to be in the negative direction.

The function that gives the position (relative to the origin) of an object as a function of time is called a **position function**.

A position function is sometimes denoted $s(t)$ when used to represent height above the ground, $x(t)$ when used to represent motion along a horizontal line, or $y(t)$ when used to represent motion along a vertical line.

Rates of Change

The **average velocity** of an object in motion with position function $s(t)$ is always on a closed interval [a, b], and can be calculated as follows:

$$\frac{s(b) - s(a)}{b - a}$$

The **(instantaneous) velocity** of an object in motion with position function $s(t)$ is always at one specific t-value, and is simply $v(t) = s'(t)$ evaluated at that t-value.

When an object's velocity is positive, its movement is in the positive direction. When an object's velocity is negative, its movement is in the negative direction.

When an object's velocity is zero, the object is at rest.

When an object's velocity changes sign from positive to negative (or vice versa), the object changes direction.

Example: Rates of Change

A ball is thrown straight down from the top of a 220-foot building with an initial velocity of -22 feet per second. The height (in feet) of the ball t seconds after it is thrown can be modeled by the position function $s(t) = -16t^2 - 22t + 220$.

(a) Find the average velocity of the ball over the interval [0, 2]. Include units in your answer.

(b) Find the velocity of the ball at $t = 2$. Include units in your answer.