§2.1 The Derivative and the Tangent Line Problem

The Tangent Line Problem

The Derivative of a Function

The Derivative and the Graphing Calculator

Differentiability and Continuity

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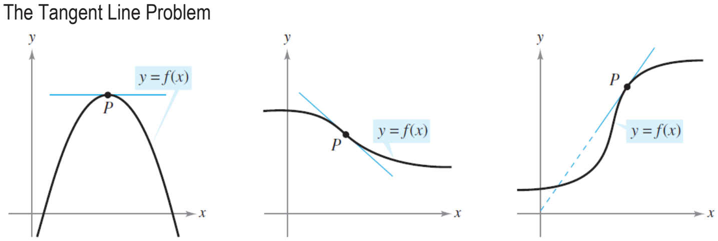
Learning Goals: Students will be able to…

- Find the slope of the tangent line to a curve at a point.
- Use the limit definition to find the derivative of a function.
- Use the alternative definition to find the derivative of a function at a point.
- Use a graphing calculator to find the derivative of a function at a point.
- Understand the relationship between differentiability and continuity.

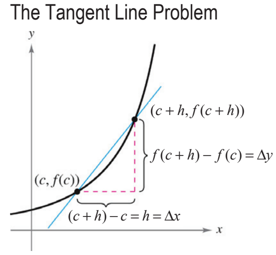
The Tangent Line Problem



A line is **tangent** to a curve at a point P when it touches or intersects the curve at exactly one point (in the given "neighborhood").

How do we find the tangent line? Essentially, this question boils down to the need to find the *slope* of the tangent line at point P.

The Tangent Line Problem



$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \left(\frac{\text{change in } y}{\text{change in } x}\right)$$

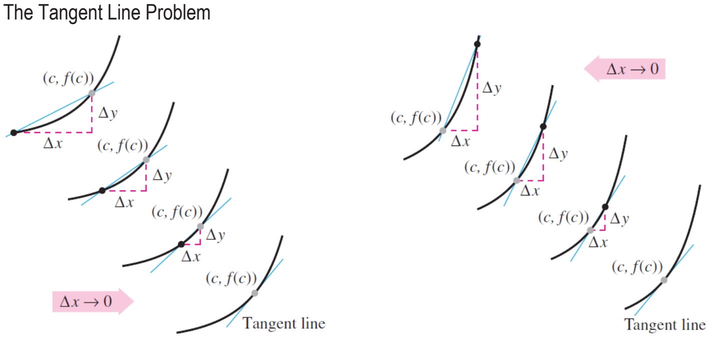
$$m_{\text{sec}} = \frac{f(c+h) - f(c)}{(c+h) - c}$$

$$m_{\text{sec}} = \frac{f(c+h) - f(c)}{h} \quad \text{(difference quotient)}$$

We can approximate this slope by using a **secant line** that goes through the point of tangency and a second point on the curve.

Point of tangency: $(c, f(c))$ Second point on the curve: $(c+h, f(c+h))$

The Tangent Line Problem



The Tangent Line Problem

DEFINITION OF TANGENT LINE WITH SLOPE *m*

If f is defined on an open interval containing c, and if the limit

$$\lim_{h \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

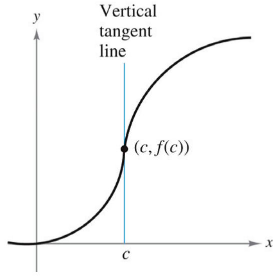
The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of f at $x = c$.**

The Tangent Line Problem

Given a point (x_1, y_1) and a slope m, an equation of the tangent line can be written in point-slope form:

$$y - y_1 = m(x - x_1)$$

If f is continuous at c and $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \infty$ or $-\infty$, then the vertical line $x = c$ passing through $(c, f(c))$ is a **vertical tangent line** to the graph of f.



The Tangent Line Problem

When attempting to find the slope of the tangent line, use of direct substitution with the limit expression $m = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ will always result in an indeterminate form $\frac{0}{0}$.

Alternate algebraic techniques must be used to find the slope.

Example: The Tangent Line Problem

Find an equation of the line tangent to the graph of $g(x) = 6 - x^2$ at (1, 5).
Use this line to approximate $g(1.3)$.

The Derivative of a Function

DEFINITION OF THE DERIVATIVE OF A FUNCTION

The **derivative** of f at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

The notation $f'(x)$ is read as "f prime of x."

The Derivative of a Function

The derivative of a function of x is also a function of x. This "new" function gives the slope of the tangent line to the graph of f at any particular point $(x, f(x))$, provided that the graph has a tangent line at that point.

The process of finding the derivative of a function is called **differentiation**.

A function is **differentiable** at x when its derivative exists at x.

A function is **differentiable on an open interval (a, b)** when it is differentiable at every point in the interval.

The Derivative of a Function

In addition to $f'(x)$, other notations are used to denote the derivative of $y = f(x)$. The most common are:

$$f'(x) \qquad \frac{dy}{dx} \qquad y' \qquad \frac{d}{dx}[f(x)] \qquad D_x[y]$$

The notation $\frac{dy}{dx}$ is read as "the derivative of y *with respect to* x" or simply "dy, dx."

The notation $\frac{d}{dx}[f(x)]$ is read as "the derivative *with respect to* x of f of x."

The *differential operator* $\frac{d}{dx}$ indicates to take the derivative with respect to x of the function $f(x)$.

Example: The Derivative of a Function

Find the derivative of $f(x) = x^3 + x^2$ using the definition of the derivative.

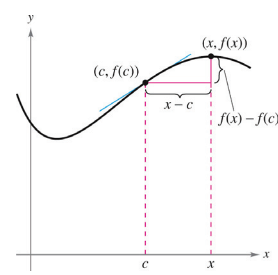
Example: The Derivative of a Function

Find the derivative of $f(x) = \dfrac{1}{x^2}$ using the definition of the derivative.

Example: The Derivative of a Function

Find the derivative of $f(x) = 4\sqrt{x}$ using the definition of the derivative.
Then find an equation of the line tangent to the graph of $f(x)$ at the point (9, 12).

The Derivative of a Function



There is also an alternative form of the definition of the derivative, which gives the derivative of a function f at c.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

With this alternative definition, the point of tangency is $(c, f(c))$ and the second point on the curve is $(x, f(x))$.

The point at x moves closer to the point of tangency—hence, x approaches c.

Since this alternative definition gives a numerical value that applies only to the point $(c, f(c))$, this value is also called a *derivative at a point*.

Example: The Derivative of a Function

Use the alternative definition of the derivative to find the derivative of $f(x) = x^3 + 6x$ at $c = 2$.

The Derivative and the Graphing Calculator

In order to succeed on the AP Calculus exam, students must be able to numerically calculate the derivative of a function.

That is, students must be able to use a graphing calculator to find a derivative at a point.

While it may be possible to find the derivative of the given function first, *then* evaluate at the given point, it is much more efficient to use the graphing calculator's "derivative at a point" feature when allowed.

Students must also be able to use a graphing calculator to find the zeros of functions. This is essential when having to solve an equation like $f'(x) = -3$.

Differentiability and Continuity

The alternative definition of the derivative is useful in investigating the relationship between differentiability and continuity.

Recall that a function $f(x)$ is differentiable at c when $f'(c)$ exists.

If using the alternative definition $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$, the existence of the limit requires that the one-sided limits $\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}$ and $\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}$ exist and are equal.

It follows that f is **differentiable on the closed interval [*a*, *b*]** when it is differentiable on (a, b) and when $\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$ and $\lim_{x\to b^-}\frac{f(x)-f(b)}{x-b}$ both exist.

Differentiability and Continuity

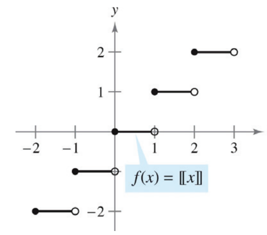
When a function is not continuous at $x=c$, it is also not differentiable at $x=c$.

For instance, the greatest integer function $f(x)=[\![x]\!]$ is not continuous at $x=0$, and so it is not differentiable at $x=0$. We can verify this by observing that:

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{[\![x]\!]-0}{x}=\infty$$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{[\![x]\!]-0}{x}=0$$

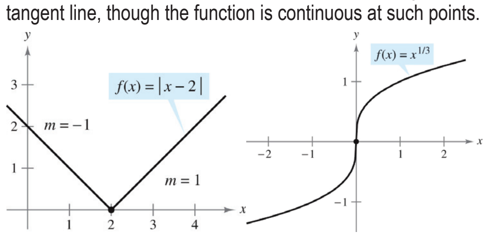
Since $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}\neq\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}$, $f'(0)$ does not exist, so $f(x)$ is not differentiable at $x=0$.



Differentiability and Continuity

When a function is continuous at $x=c$, it may or may not be differentiable at $x=c$.

A function is not differentiable at a point at which its graph has a sharp turn *or* a vertical tangent line, though the function is continuous at such points.



Differentiability and Continuity

THEOREM DIFFERENTIABILITY IMPLIES CONTINUITY

If f is differentiable at $x = c$, then f is continuous at $x = c$.

This implication is important for theorems that require continuity, such as the Intermediate Value Theorem.

Given a statement that a function is differentiable on an interval, we automatically know it is also continuous on the same interval.