§1.6 Limits at Infinity

Limits at Infinity

Horizontal Asymptotes

Relative Magnitudes of Functions

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.
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Learning Goals: Students will be able to…

- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

Limits at Infinity

Given $f(x)=\dfrac{3x^2}{x^2+1}$, find the limits $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$ numerically and graphically. Explain your reasoning.

← *x* decreases without bound. | *x* increases without bound. →

x	−1000	−100	−10	0	10	100	1000
$f(x)$	2.9999	2.9997	2.97	0	2.97	2.9997	2.9999

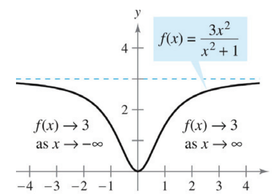
← $f(x)$ approaches 3. | $f(x)$ approaches 3. →

Limits at Infinity

Given $f(x)=\dfrac{3x^2}{x^2+1}$, find the limits $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$ numerically and graphically. Explain your reasoning.

$\lim\limits_{x\to-\infty} f(x)=3$. As x decreases without bound, $f(x)$ approaches 3.

$\lim\limits_{x\to\infty} f(x)=3$. As x increases without bound, $f(x)$ approaches 3.



Horizontal Asymptotes

DEFINITION OF A HORIZONTAL ASYMPTOTE

The line $y = L$ is a **horizontal asymptote** of the graph of f if

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L.$$

When justifying the existence of a horizontal asymptote, only one of the limits at infinity is required.

The graph of a function f can have at most two horizontal asymptotes—one justified by $\lim\limits_{x\to-\infty} f(x)$, and another justified by $\lim\limits_{x\to\infty} f(x)$.

Rational functions—that is, a *polynomial* divided by another *polynomial*—can have only one horizontal asymptote. Either limit at infinity can be used as justification.

Horizontal Asymptotes

THEOREM LIMITS AT INFINITY

If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0.$$

Example: Horizontal Asymptotes

Given $f(x)=\dfrac{3-2x}{3x^3-1}$, find the limit $\lim_{x\to\infty} f(x)$ algebraically. Use the result to identify any horizontal asymptotes of the graph of $f(x)$.

Example: Horizontal Asymptotes

Given $f(x)=\dfrac{3-2x}{3x-1}$, find the limit $\lim_{x\to\infty} f(x)$ algebraically. Use the result to identify any horizontal asymptotes of the graph of $f(x)$.

Example: Horizontal Asymptotes

Given $f(x)=\dfrac{3-2x^2}{3x-1}$, find the limit $\lim_{x\to\infty} f(x)$ algebraically. Use the result to identify any horizontal asymptotes of the graph of $f(x)$.

Example: Horizontal Asymptotes

Given $f(x)=\dfrac{x-4}{\sqrt{x^2+1}}$, find the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ algebraically. Use the result to identify any horizontal asymptotes of the graph of $f(x)$.

Example: Horizontal Asymptotes (cont.)

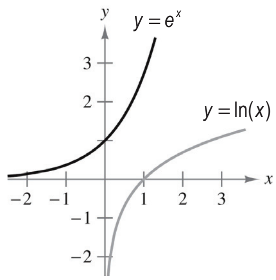
Given $f(x)=\dfrac{x-4}{\sqrt{x^2+1}}$, find the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ algebraically. Use the result to identify any horizontal asymptotes of the graph of $f(x)$.

Horizontal Asymptotes

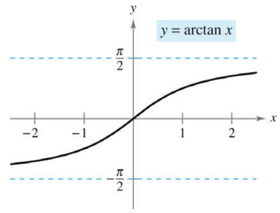
End behavior of transcendental functions is also important when solving limits at infinity and finding horizontal asymptotes.

$\lim_{x\to-\infty} e^x = 0$ $\qquad \lim_{x\to0^+} \ln(x) = -\infty$

$\lim_{x\to\infty} e^x = \infty$ $\qquad \lim_{x\to\infty} \ln(x) = \infty$

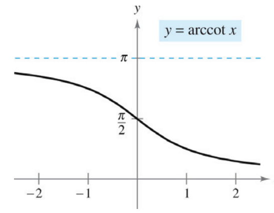


Horizontal Asymptotes



$\lim_{x\to-\infty} \arctan(x) = -\pi/2$

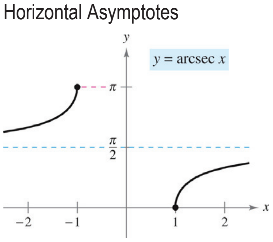
$\lim_{x\to\infty} \arctan(x) = \pi/2$



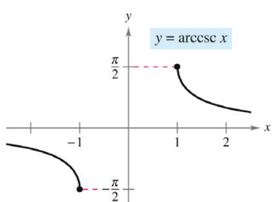
$\lim_{x\to-\infty} \operatorname{arccot}(x) = \pi$

$\lim_{x\to\infty} \operatorname{arccot}(x) = 0$

Horizontal Asymptotes



$\lim_{x\to-\infty} \operatorname{arcsec}(x) = \lim_{x\to\infty} \operatorname{arcsec}(x) = \pi/2$



$\lim_{x\to-\infty} \operatorname{arccsc}(x) = \lim_{x\to\infty} \operatorname{arccsc}(x) = 0$

Example: Horizontal Asymptotes

Given $f(x) = \dfrac{6}{2+e^x}$, find the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ algebraically. Use the result to identify any horizontal asymptotes of the graph of $f(x)$.

Relative Magnitudes of Functions

As x increases without bound, different classes of functions increase more rapidly than others. They rank, in order from fastest to slowest:

1. Exponential functions, such as $f(x) = e^x$
2. Polynomial functions, such as $f(x) = x^r$
3. Logarithmic functions, such as $f(x) = \ln(x)$

This can help when we have limits at infinity. Functions that increase more rapidly "dominate" functions that grow at a slower rate.

Example: Relative Magnitudes of Functions

Find the limits $\lim_{x\to\infty} \dfrac{x^3}{e^{4x}}$ and $\lim_{x\to\infty} \dfrac{x^5}{\ln(4x)}$. Explain your reasoning using relative magnitudes.