

**§1.6 Limits at Infinity**

Limits at Infinity  
Horizontal Asymptotes  
Relative Magnitudes of Functions

Notes based on: *Calculus for AP* by Larson & Battaglia. © 2017 Cengage Learning.  
*Calculus, AP Edition, 9th ed.* by Larson & Edwards. © 2010 Brooks/Cole, Cengage Learning.

**Learning Goals: Students will be able to...**

- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

**Learning Objectives: Students will be able to...**

- 1.1A Express limits symbolically using correct notation, and interpret limits expressed symbolically.
- 1.1B Estimate limits of functions.
- 1.1C Determine limits of functions.
- 1.1D Deduce and interpret behavior of functions using limits.

Limits at Infinity

Given  $f(x) = \frac{3x^2}{x^2 + 1}$ , find the limits  $\lim_{x \rightarrow -\infty} f(x)$  and  $\lim_{x \rightarrow \infty} f(x)$  numerically and graphically.

Explain your reasoning.

←  $x$  decreases without bound.       $x$  increases without bound. →

$x$	- 1000	- 100	- 10	0	10	100	1000
$f(x)$	2.9999	2.9997	2.97	0	2.97	2.9997	2.9999

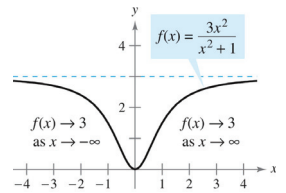
←  $f(x)$  approaches 3.       $f(x)$  approaches 3. →

Limits at Infinity

Given  $f(x) = \frac{3x^2}{x^2 + 1}$ , find the limits  $\lim_{x \rightarrow -\infty} f(x)$  and  $\lim_{x \rightarrow \infty} f(x)$  numerically and graphically.

Explain your reasoning.

$\lim_{x \rightarrow -\infty} f(x) = 3$ . As  $x$  decreases without bound,  $f(x)$  approaches 3.  
 $\lim_{x \rightarrow \infty} f(x) = 3$ . As  $x$  increases without bound,  $f(x)$  approaches 3.



Horizontal Asymptotes

**DEFINITION OF A HORIZONTAL ASYMPTOTE**

The line  $y = L$  is a **horizontal asymptote** of the graph of  $f$  if

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow \infty} f(x) = L.$$

When justifying the existence of a horizontal asymptote, only one of the limits at infinity is required.

The graph of a function  $f$  can have at most two horizontal asymptotes—one justified by  $\lim_{x \rightarrow -\infty} f(x)$ , and another justified by  $\lim_{x \rightarrow \infty} f(x)$ .

Rational functions—that is, a *polynomial* divided by another *polynomial*—can have only one horizontal asymptote. Either limit at infinity can be used as justification.

Horizontal Asymptotes

**THEOREM      LIMITS AT INFINITY**

If  $r$  is a positive rational number and  $c$  is any real number, then

$$\lim_{x \rightarrow \infty} \frac{c}{x^r} = 0.$$

Furthermore, if  $x^r$  is defined when  $x < 0$ , then

$$\lim_{x \rightarrow -\infty} \frac{c}{x^r} = 0.$$

Example: Horizontal Asymptotes

Given  $f(x) = \frac{3-2x}{3x^3-1}$ , find the limit  $\lim_{x \rightarrow \infty} f(x)$  algebraically. Use the result to identify any horizontal asymptotes of the graph of  $f(x)$ .

Example: Horizontal Asymptotes

Given  $f(x) = \frac{3-2x}{3x-1}$ , find the limit  $\lim_{x \rightarrow \infty} f(x)$  algebraically. Use the result to identify any horizontal asymptotes of the graph of  $f(x)$ .

Example: Horizontal Asymptotes

Given  $f(x) = \frac{3-2x^2}{3x-1}$ , find the limit  $\lim_{x \rightarrow \infty} f(x)$  algebraically. Use the result to identify any horizontal asymptotes of the graph of  $f(x)$ .

Example: Horizontal Asymptotes

Given  $f(x) = \frac{x-4}{\sqrt{x^2+1}}$ , find the limits  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow -\infty} f(x)$  algebraically. Use the result to identify any horizontal asymptotes of the graph of  $f(x)$ .

Example: Horizontal Asymptotes (cont.)

Given  $f(x) = \frac{x-4}{\sqrt{x^2+1}}$ , find the limits  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow -\infty} f(x)$  algebraically. Use the result to identify any horizontal asymptotes of the graph of  $f(x)$ .

Horizontal Asymptotes

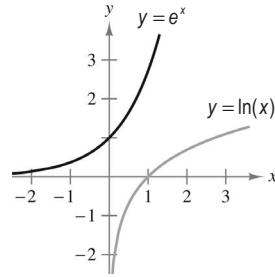
End behavior of transcendental functions is also important when solving limits at infinity and finding horizontal asymptotes.

$$\lim_{x \rightarrow -\infty} e^x = 0$$

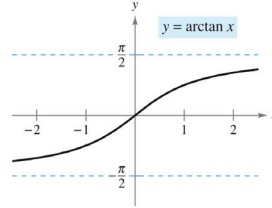
$$\lim_{x \rightarrow 0^+} \ln(x) = -\infty$$

$$\lim_{x \rightarrow \infty} e^x = \infty$$

$$\lim_{x \rightarrow \infty} \ln(x) = \infty$$

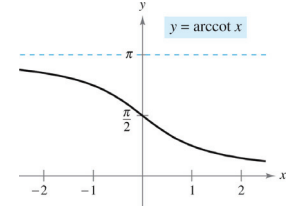


Horizontal Asymptotes



$$\lim_{x \rightarrow -\infty} \arctan(x) = -\pi/2$$

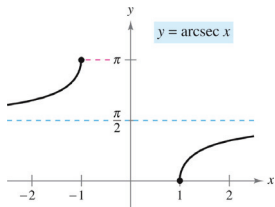
$$\lim_{x \rightarrow \infty} \arctan(x) = \pi/2$$



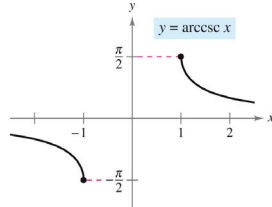
$$\lim_{x \rightarrow -\infty} \operatorname{arccot}(x) = \pi$$

$$\lim_{x \rightarrow \infty} \operatorname{arccot}(x) = 0$$

Horizontal Asymptotes



$$\lim_{x \rightarrow -\infty} \operatorname{arcsec}(x) = \lim_{x \rightarrow \infty} \operatorname{arcsec}(x) = \pi/2$$



$$\lim_{x \rightarrow -\infty} \operatorname{arccsc}(x) = \lim_{x \rightarrow \infty} \operatorname{arccsc}(x) = 0$$

Example: Horizontal Asymptotes

Given  $f(x) = \frac{6}{2 + e^x}$ , find the limits  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow -\infty} f(x)$  algebraically. Use the result to identify any horizontal asymptotes of the graph of  $f(x)$ .

Relative Magnitudes of Functions

As  $x$  increases without bound, different classes of functions increase more rapidly than others. They rank, in order from fastest to slowest:

1. Exponential functions, such as  $f(x) = e^x$
2. Polynomial functions, such as  $f(x) = x^r$
3. Logarithmic functions, such as  $f(x) = \ln(x)$

This can help when we have limits at infinity. Functions that increase more rapidly "dominate" functions that grow at a slower rate.

Example: Relative Magnitudes of Functions

Find the limits  $\lim_{x \rightarrow \infty} \frac{x^3}{e^{4x}}$  and  $\lim_{x \rightarrow \infty} \frac{x^5}{\ln(4x)}$ . Explain your reasoning using relative magnitudes.